Tidal effects in eccentric compact binaries

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February 16^{th} 2026

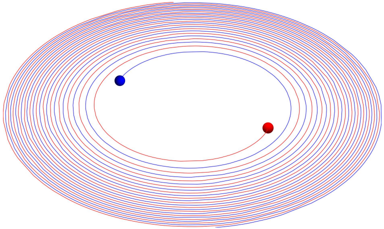
Based on several works

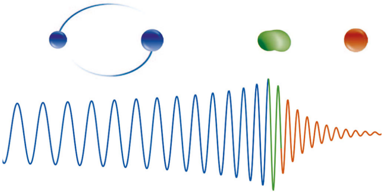
- [HFB19] Q.H., G. Faye, L. Blanchet arXiv:1912.01920
- [HFB20a] Q.H., G. Faye, L. Blanchet arXiv:2005.13367
- [HFB20b] Q.H., G. Faye, L. Blanchet arXiv:2009.12332
- [DHB24] E. Dones, Q.H., L. Bernard arXiv:2412.14249
- [HH25] Q.H., A. Heffernan arXiv:2512.06489
- [H26] Q.H. arXiv:2601.01794

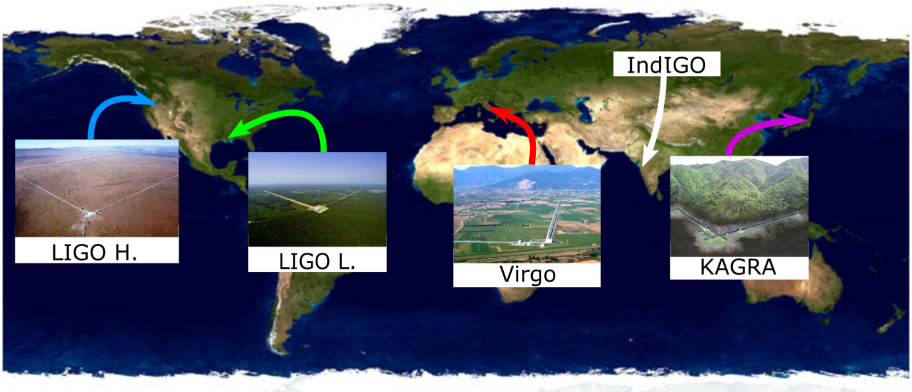
Outline

1. Introduction
   - Detections and physical implications
   - Waveform modeling
2. The effective action
3. Dynamics
   - The conservative dynamics: quasi-Keplerian parametrisation
   - The radiative dynamics
4. Full gravitational waveform
   - Fluxes and secular evolution
   - Amplitude modes
5. Conclusion

Outline






IndIGO
LIGO H.
LIGO L.
Virgo
KAGRA

Detections

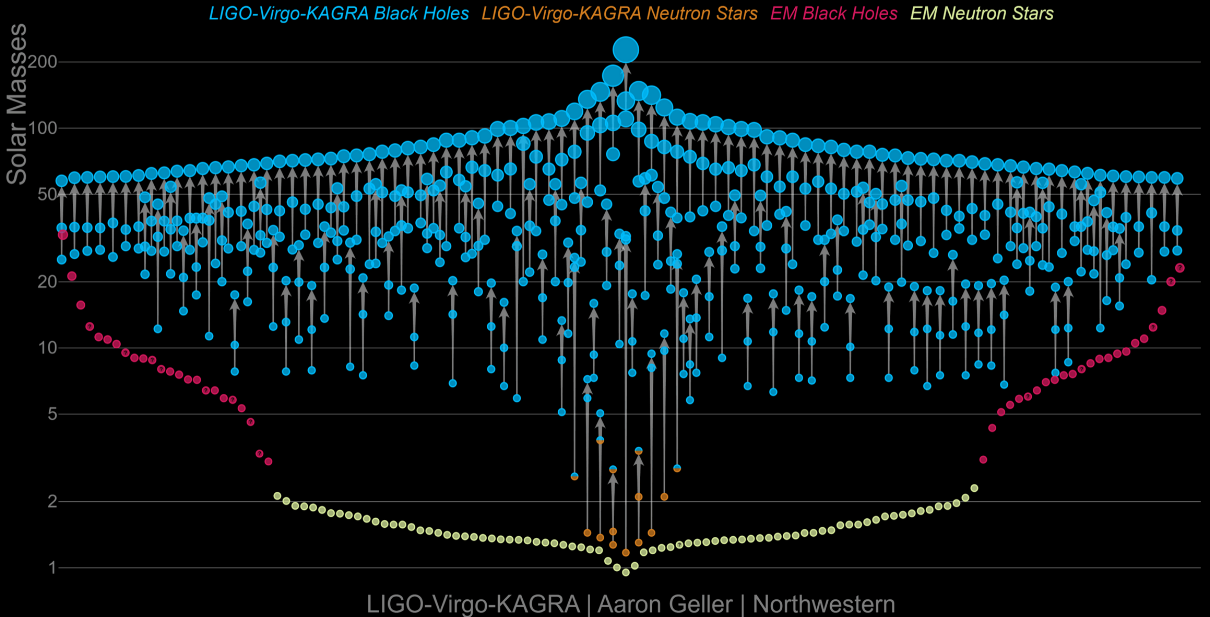
Observational status

- Four observing runs of the LVK network: O1, O2, O3, O4
- **GWTC–4.0:** 218 gravitational-wave candidates
  - predominantly binary black hole (BBH) mergers
  - binary neutron star (BNS): 2 or 3
  - neutron star–black hole (NSBH): 5 or 6
  - Compact objects in the low mass gap (2-5$M_\odot$): 3 or 4 very massive NS ? low mass BH ?

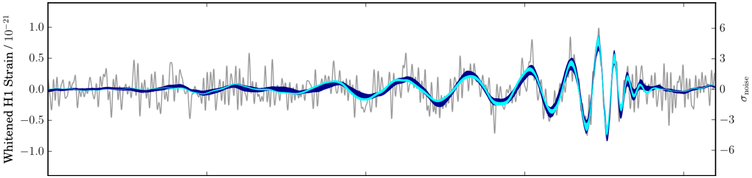
Notable events

- **GW150914**: first direct detection (BBH)
- **GW170817**: first BNS detection (only one with electromagnetic counterpart)
- **GW200105**: NSBH on eccentric orbit
- **GW231123**: most massive, debated interpretation (high mass gap? probably lensed?)
  - $m_1 = 137^{+22}_{-17} M_\odot$, $m_2 = 103^{+20}_{-52} M_\odot$, $m_f = 225^{+26}_{-43} M_\odot$

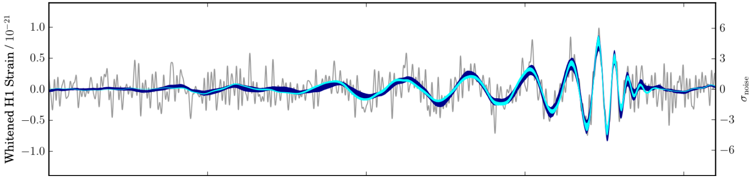
Masses in the Stellar Graveyard



GW150914

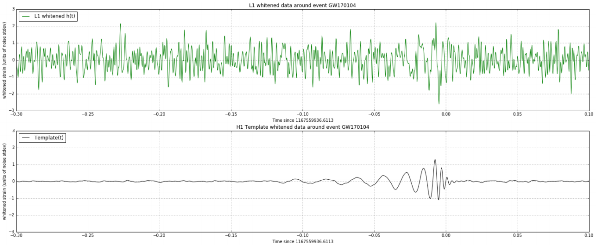


GW150914



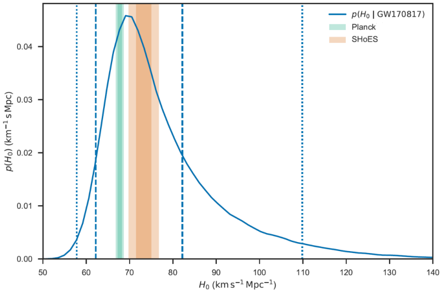
GW170104

- Need **precise templates** to extract the signal from the noise
- We can **infer the properties** of the companions (mass, spin...) using matched filtering

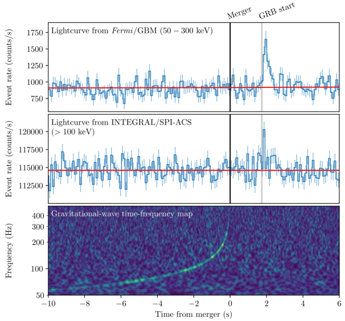


Focusing on BNS mergers: what can we learn ?

- **Population studies:** NS formation channels, Active Galactic Nuclei
- Constrain the **EOS of neutron stars**
- Independent measurement of the Hubble-Lemaître constant H_0



- **Multimessenger:**
  - nuclear astrophysics (origin of heavy elements)
  - GW170817 : $d_L \sim 40Mpc$



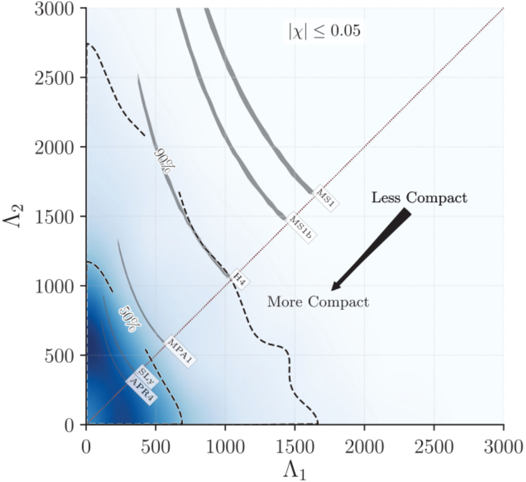
  - Constrain on $|c_g - c_{em}| \lesssim 10^{-15} c_{em}$
  - Discarded alternative theories of gravity

The constrain on EOS from GW170817

Tidal deformability parameter

$$\Lambda = \frac{2k^{(2)}}{3}\left(\frac{c^2 R}{Gm}\right)^5$$

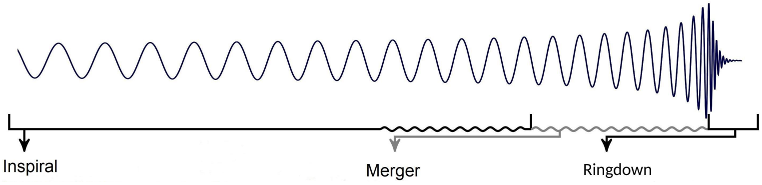
$k^{(2)}$: second Love number, parametrizes quadrupolar deformation



[LIGO, Virgo PRL 116, 061102 (2016)]

Strain

$$h_+(t) - ih_\times(t) = \sum_{\ell\geq 2}\sum_{|m|\leq\ell} {}_{-2}Y_{\ell m}\, h_{\ell m}(t)\, e^{-im\psi(t)}$$



Inspiral

- High number of cycles
- Need analytical model
- PN, GSF, PM...

Merger

- EE highly non-linear
- Only solvable with NR
- Mostly short simulations

Ringdown

- Final object relaxing
- Quasi-normal modes
- Complex for NS

Key takeaway

Waveform models combine the three phases : full inspiral-merger-ringdown description

Waveform model families

Effective-One-Body (EOB)

- Semi-analytical : mapping to an effective-one-body problem matched to NR
- Very precise, slower to produce templates : high precision data analysis

Phenomenological (Phenom)

- Semi-analytical : PN dynamics & waveform, phenomenology (phasing), matched to NR
- Fast template production, less accurate : relevant in searches, also used in data analysis

Numerical relativity surrogate (NRS)

- Extrapolation of NR simulations, some models are hybridized with PN
- Most precise for short signals, depends on available NR simulations

Key takeaway

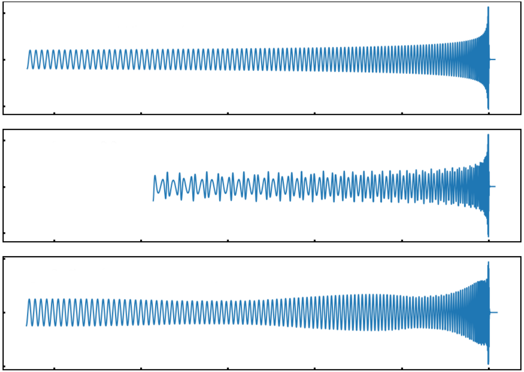
(Almost) All models incorporate PN results for the modeling of the inspiral

Why include physical effects in waveform models?

Circularized point-mass waveforms describe isolated BBH, but $\sim 5-10\%$ of sources are expected to show **detectable deviations** (eccentricity, spin) in LVK [Zevin 2021].

Beyond the point-mass approximation

- **Compact-object physics**
  - spins, tides, BH absorption
- **Environmental effects**
  - accretion disk, mass transfer
  - magnetic fields, triple systems



Waveforms with eccentricity or spin precession.

Key takeaway

Analyzing complex signals with simplified templates leads to huge biases in parameter inference

The post-Newtonian formalism & tidal effects

Post-Newtonian formalism

- Perturbative expansion of the equations of GR (Minkowski background)

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu} \quad \text{with} \quad \tau^{\mu\nu} = |g|T^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda^{\mu\nu}(h, \partial h, \partial^2 h)$$

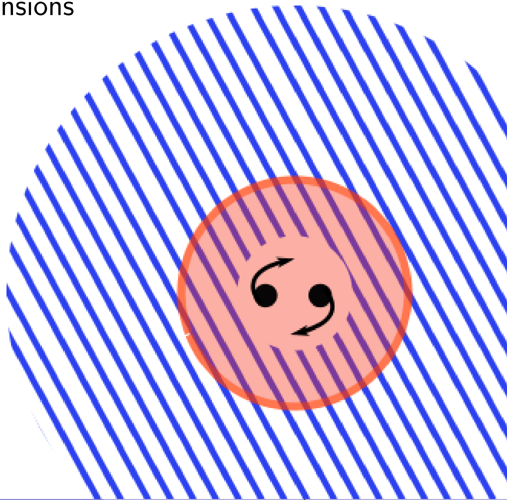
- Small velocities, weak field: $(v/c)^2 \sim GM/rc^2 \ll 1$
- nPN order $\rightarrow \mathcal{O}(1/c^{2n})$ beyond leading order $\Rightarrow$ 2.5PN $= \mathcal{O}(1/c^5)$
- Can be applied to any matter action
- Best approach to model the inspiral phase of comparable-mass compact binaries

Tidal effects in PN:

- low mass $\Rightarrow$ high number of cycles (GW170817: ~ 3000) more than NR simulations
- Late inspiral effect (no additional frequency for adiabatic tides)

The PN-MPM formalism in a nutshell

- Near zone: PN expansion $(v/c \ll 1)$, multipole expansion matched the source
- Exterior zone: PM expansion $(G \ll 1)$, **source moments** $\{I_L, J_L\}$
- Buffer zone: matching PN and PM multipole expansions
- Radiative zone: **radiative moments** $\{U_L, V_L\}$



In practice, we perform two computations

- **Conservative sector**
  - $\hookrightarrow$ Conservative dynamics
  - $\hookrightarrow$ Conserved quantities
- **Radiative sector**
  - $\hookrightarrow$ Radiated fluxes, radiative dynamics
  - $\hookrightarrow$ Observables: phase and amplitude of the GW

State of the art for adiabatic tides within PN

Gravitational wave observables: All consistent to relative 2.5PN

- Quasi-circular energy flux, GW phase ψ [HFB20a]
- Complete quasi-circular PN-expanded & EOB-factorized amplitude modes [DHB24]

$$h_{\ell m}^{\text{EOB}} = h_{\ell m}^{\text{N}} S_{\text{eff}} T_{\ell m} f_{\ell m} e^{i\delta_{\ell m}}$$

- No results for eccentric orbits

Implemented in every waveform models describing BNS/NSBH

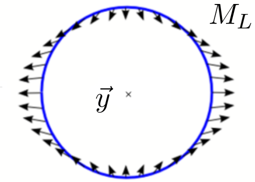
- EOB
  - `TEOBResumS` [Gamba et al. 2022]
  - `SEOBNRv5THM` [Haberland et al. 2025]
- Phenom
  - `NRTidalv3` [Abac et al. 2023]
  - `IMRPhenomXPHM_NSBH` [To be published]

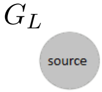
Outline

Tides in Newtonian gravity

Basics:

- An extended body in a non-homogeneous gravitational field gets deformed
- The deformability of the body is parametrized with **Love numbers** $k^{(\ell)}$





$$\boxed{F^i_{\text{ext}} = M\partial_i U_{\text{ext}} + \sum_{\ell\geq 2} \frac{1}{\ell!} M_L G_{iL}}$$

Two types of tidal effects

- **Adiabatic :** the extended body is at thermodynamical equilibrium at each instant
  - Matter moments aligned on tidal field : $\boxed{M_L = \mu^{(\ell)} G_L}$
- **Dynamical :** vibration modes can enter in resonance with the orbital frequency

Effective action for spinless adiabatic finite-size effects in GR

Most general non-minimal worldline action satisfying:

- depending only on the metric $g_{\mu\nu}$ and 4-velocity u^μ
- invariance under reparametrisation
- parity invariance

$$S_{\text{nm}} = \sum_A \sum_{\ell\geq 2} \frac{1}{2\ell!} \Bigg[\mu_A^{(\ell)} \int \mathrm{d}\tau_A G^A_{\alpha_1\dots\alpha_\ell}(\tau_A) G_A^{\alpha_1\dots\alpha_\ell}(\tau_A) + \frac{\ell\,\sigma_A^{(\ell)}}{(\ell+1)c^2} \int \mathrm{d}\tau_A\, H^A_{\alpha_1\dots\alpha_\ell}(\tau_A) H_A^{\alpha_1\dots\alpha_\ell}(\tau_A)$$
$$+ \frac{{\mu'}_A^{(\ell)}}{c^2} \int \mathrm{d}\tau_A \dot{G}^A_{\alpha_1\dots\alpha_\ell}(\tau_A) \dot{G}_A^{\alpha_1\dots\alpha_\ell}(\tau_A) + \frac{\ell\,{\sigma'}_A^{(\ell)}}{(\ell+1)c^4} \int \mathrm{d}\tau_A \dot{H}^A_{\alpha_1\dots\alpha_\ell}(\tau_A) \dot{H}_A^{\alpha_1\dots\alpha_\ell}(\tau_A) + \dots \Bigg]$$

- G and H are the electric and magnetic parts of the Weyl tensor (contracted with u^μ)
- Contains also higher order terms (e.g. cubic)
- Regularised on the location of body A [Bini, Damour, Faye 2012]

Effective action at 2PN

Total action:

$$\boxed{S = S_{\mathsf{EH}} + S_{\mathsf{pp}} + S_{\mathsf{T}}}$$

$$S_{\mathsf{T}} = \sum_{\mathsf{A}=1,2} \int \mathrm{d}\tau_A \left[\frac{\mu_A^{(2)}}{4} G^A_{\alpha\beta} G_A^{\alpha\beta} + \frac{\sigma_A^{(2)}}{6c^2} H^A_{\alpha\beta} H_A^{\alpha\beta} + \frac{\mu_A^{(3)}}{12} G^A_{\alpha\beta\gamma} G_A^{\alpha\beta\gamma} \right]$$

Tidal tensors

- $G_{\alpha\beta} = -R_{\alpha\mu\beta\nu} u^\mu u^\nu$: mass quadrupole
- $H_{\alpha\beta} = 2R^*_{\alpha\mu\beta\nu} u^\mu u^\nu$: current quadrupole
- $G_{\alpha\beta\gamma}$: mass octupole

Effective action at 2PN

$$S_{\mathsf{T}} = \sum_{\mathsf{A}=1,2} \int \mathrm{d}\tau_A \left[\frac{\mu_A^{(2)}}{4} G^A_{\alpha\beta} G_A^{\alpha\beta} + \frac{\sigma_A^{(2)}}{6c^2} H^A_{\alpha\beta} H_A^{\alpha\beta} + \frac{\mu_A^{(3)}}{12} G^A_{\alpha\beta\gamma} G_A^{\alpha\beta\gamma} \right]$$

Tidal polarizabilities

- $\mu^{(\ell)}$ and $\sigma^{(\ell)}$ linked to Love numbers : $\mu_A^{(\ell)} \propto k_A^{(\ell)} R_A^{2\ell+1}$.
- $\mathcal{C} = \dfrac{Gm}{Rc^2} \sim 1$ for compact objects.
- $\mu^{(2)} \sim \mathcal{O}\left(\dfrac{1}{c^{10}}\right) \sim \sigma^{(2)} \rightarrow$ Leading order in the PN expansion (5PN)
- $\mu^{(3)} \sim \mathcal{O}\left(\dfrac{1}{c^{14}}\right) \rightarrow$ relative 2PN (7PN)

Outline

Overview of intermediate computations

1) PN Lagrangian/Hamiltonian [HFB19, HFB20b, DHB24]

- Compute the Fokker action and associated reduced Lagrangian & Hamiltonian
  - 3+1 decomposition
  - Insert the off-shell PN metric (potentials)
  - Reduction
- Deduce the EOM and the 10 conserved quantities $\{E, J^i, P^i, G^i\}$ in harmonic gauge

2) Conservative dymamics [HH25]

- Quasi-Keplerian parametrisation : in terms of the conserved quantities
- $(r, \dot{r}, \phi, \dot{\phi})$ in terms of (x, e_t, u)
- $(r, \dot{r}, \phi, \dot{\phi})$ in terms of $(x, e_t, l) \rightarrow$ inversion of the generalized Kepler equation

3) Radiative dynamics : the 2.5PN radiation reaction terms [DHB24, HH25]

- Allow time dependency on the constants of motion : variation of the constants method

Solving for the conservative dynamics

- Start from conserved quantities (NNLO)

$$\widetilde{E} = \frac{\dot{r}^2}{2} + \frac{r^2\dot{\phi}^2}{2} - \frac{GM}{r} - \frac{3\,G^2M}{r^6}\mu_+^{(2)}$$

$$h = \frac{r^2\dot{\phi}}{GM}$$

- Inverse iteratively with $s = \frac{1}{r}$

$$\dot{r}^2 = \sum_{k=0}^{10} p_k(\widetilde{E},h)\, s^k = \mathcal{P}(s)$$

$$\dot{\phi} = \sum_{k=2}^{10} q_k(\widetilde{E},h)\, s^k = s^2\mathcal{Q}(s)$$

- Ansatz : $r = a_r(1 - e_r\cos u)$
- s goes through a max and min

$$\mathcal{P}(s) = (s_+ - s)(s - s_-)\mathcal{R}(s)$$

- Inject in $\dot{r}$ and $\dot{\phi}$

$$t - t_0 = \int_s^{s_+} \frac{[\mathcal{R}(x)]^{-1/2}}{x^2\sqrt{(s_+ - x)(x - s_-)}}\,\mathrm{d}x$$

$$\phi - \phi_0 = \int_s^{s_+} \frac{\mathcal{Q}(x)\,[\mathcal{R}(x)]^{-1/2}}{\sqrt{(s_+ - x)(x - s_-)}}\,\mathrm{d}x$$

- Kernel integrals, derived $\forall n \in \mathbb{N}$,

$$I_n = \int_s^{s_+} \frac{x^{n-2}}{\sqrt{(s_+ - x)(x - s_-)}}\,\mathrm{d}x$$

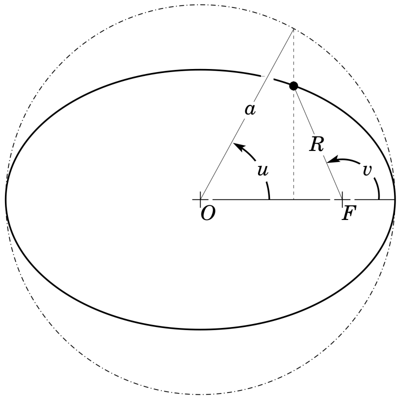
Quasi-Keplerian parametrisation : in terms of the conserved quantities

$$r = a_r(1 - e_r \cos u)$$

$$l = n(t - t_0) = u - e_t \sin u + f_{v-u}(v - u) + \sum_{k=1}^{6} f_{kv} \sin(kv)$$

$$\frac{\phi - \phi_0}{K} = v + \sum_{k=2}^{8} g_{kv} \sin(kv)$$

$$v = 2 \arctan\left[\sqrt{\frac{1 + e_\phi}{1 - e_\phi}} \tan\frac{u}{2}\right]$$



The parameters

- e_r, e_ϕ, e_t : eccentric parameters
- K : periastron advance
- n : mean motion ; l : mean anomaly
- u : eccentric anomaly ; v : true anomaly

Conservative dynamics : $(r, \dot{r}, \phi, \dot{\phi})$ in terms of (x, e_t, u)

- Define the dimensionless PN parameter

$$x = \left(\frac{GM\Omega}{c^3}\right)^{2/3}$$

$\Omega = Kn$: orbital frequency

- Invert the system

$$x\,c^2 = -2\widetilde{E} + f(\widetilde{E}, h)$$
$$e_t^2 = 1 + 2\widetilde{E}h^2 + g(\widetilde{E}, h)$$

- Use chain rule on r and ϕ

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{n}{\mathrm{d}l/\,\mathrm{d}u}\frac{\mathrm{d}}{\mathrm{d}u}$$

- Replace $\widetilde{E}$ and h in $(r, \dot{r}, \phi, \dot{\phi})$
- Deduce conservative dynamics in terms of (x, e_t, u)

Conserved quantities as functions of (x, e_t)

$$-\widetilde{E} = \frac{c^2 x}{2}\left[1 - \frac{3}{4}\frac{x^5\widetilde{\mu}_+^{(2)}}{(1-e_t^2)^5}\left(5(8+12e_t^2+e_t^4) - 4(1-e_t^2)^{3/2}(4+e_t^2)\right)\right] + \mathcal{O}\left(\frac{1}{c^2}\right)$$

$$h = \frac{\sqrt{1-e_t^2}}{c\sqrt{x}}\left[1 + \frac{3\widetilde{\mu}_+^{(2)}x^5}{8(1-e_t^2)^5}\left(48 + 108e_t^2 + 13e_t^4 - 4\sqrt{1-e_t^2}\left(4+9e_t^2+2e_t^4\right)\right)\right] + \mathcal{O}\left(\frac{1}{c^2}\right)$$

Conservative dynamics : $(r, \dot{r}, \phi, \dot{\phi})$ in terms of (x, e_t, u)

$$r = \frac{GM}{x\,c^2}\left\{1 - e_t\cos u + \frac{3}{4}\frac{x^5\widetilde{\mu}^{(2)}_+}{(1-e_t^2)^5}\left[3(8+20e_t^2+7e_t^4) - 4(1-e_t^2)^{3/2}(4+e_t^2) - e_t\left(5(8+12e_t^2+e_t^4) + 2(1-e_t^2)^{3/2}(4+e_t^2)\right)\cos u\right]\right\}$$

$$\begin{aligned}\dot{r} =& \frac{c\sqrt{x}\,e_t\sin u}{1-e_t\cos u} - 3\frac{e_t\,c\,x^{11/2}\widetilde{\mu}^{(2)}_+\sin u}{(1-e_t^2)^5}\left\{\frac{(1-e_t^2)^4}{(1-e_t\cos u)^5} + \frac{2(1-e_t^2)^3}{(1-e_t\cos u)^4} + \frac{(1-e_t^2)^2(3+e_t^2)}{(1-e_t\cos u)^3}\right.\\ &\left. - \frac{3(1-e_t^2)^{3/2}(4+e_t^2)}{2(1-e_t\cos u)^2} + \frac{5(8+12e_t^2+e_t^4) - 4(1-e_t^2)^{3/2}(4+e_t^2)}{8(1-e_t\cos u)}\right\} + \mathcal{O}\left(\frac{1}{c^2}\right)\end{aligned}$$

$$\begin{aligned}\phi =& 2\arctan\left[\sqrt{\frac{1+e_t}{1-e_t}}\tan\frac{u}{2}\right]\left(1 + \frac{45(8+12e_t^2+e_t^4)}{8(1-e_t^2)^5}\widetilde{\mu}^{(2)}_+x^5\right)\\ &+ \frac{3e_t\widetilde{\mu}^{(2)}_+x^5\sin u}{4(1-e_t^2)^{9/2}}\left\{\frac{(1-e_t^2)^3}{(1-e_t\cos u)^4} + \frac{5(1-e_t^2)^2}{(1-e_t\cos u)^3} + \frac{34-27e_t^2-7e_t^4}{2(1-e_t\cos u)^2} + \frac{146+105e_t^2+12\sqrt{1-e_t^2}(4+e_t^2)}{2(1-e_t\cos u)}\right\} + \mathcal{O}\left(\frac{1}{c^2}\right)\end{aligned}$$

$$\begin{aligned}\dot{\phi} =& \frac{x^{3/2}c^3}{GM}\left\{\frac{\sqrt{1-e_t^2}}{(1-e_t\cos u)^2} + \frac{\widetilde{\mu}^{(2)}_+x^5}{(1-e_t^2)^5}\left[\frac{9(1-e_t^2)^2(4+e_t^2) + 24\sqrt{1-e_t^2}(1-e_t^4)}{(1-e_t\cos u)^3}\right.\right.\\ &\left.\left. - \frac{36(1-e_t^2)(4+e_t^2) + 3\sqrt{1-e_t^2}(112+132e_t^2+7e_t^4)}{8(1-e_t\cos u)^2}\right]\right\} + \mathcal{O}\left(\frac{1}{c^2}\right)\end{aligned}$$

Conservative dynamics : $(r, \dot{r}, \phi, \dot{\phi})$ in terms of (x, e_t, l)

Inversion of the generalized Kepler equation

$$l = u - e_t \sin u + f_{v-u}(v-u) + \sum_{k=1}^{6} f_{kv} \sin(kv)$$

Fourier series:

$$u - l = \sum_{k=1}^{\infty} A_k(x, e_t) \sin(kl)$$

$$A_k = \frac{2}{k} J_k(ke_t) + \sum_{j=1}^{\infty} \alpha_j \Big(J_{k+j}(ke_t) - J_{k-j}(ke_t) \Big)$$
$$+ \frac{k}{4} \sum_{j=1}^{\infty} \sum_{i=1}^{j} \alpha_i \alpha_{j-i+1} \Big(J_{k+j+1}(ke_t) + J_{k-j-1}(ke_t) - J_{k+2i-j-1}(ke_t) - J_{k-2i+j+1}(ke_t) \Big) + \mathcal{O}\big(\alpha_j^3\big)$$

- α_j are functions of (x, e_t)
- Need quadratic order in α_j (not required for point masses)
- After excentricity expansion, $e_t < e_{\text{max}} \simeq 0.6627$

Conservative dynamics : $(r, \dot{r}, \phi, \dot{\phi})$ in terms of (x, e_t, l)

- **Derived $u(l)$ at NNLO** to $\mathcal{O}(e_t^{14})$
  - Eccentricity order motivated by Phenom and EOB models

- **Deduced $(r, \dot{r}, \phi, \dot{\phi})$ at NNLO** in terms of (x, e_t, l)

$$r = \frac{GM}{x\,c^2}\left[1 + \frac{e_t^2}{2} + \widetilde{\mu}_+^{(2)} x^5\left(6 + 93 e_t^2\right) - e_t\cos(l) - \frac{e_t^2}{2}\cos(2l)\left(1 + 6\widetilde{\mu}_+^{(2)} x^5\right)\right] + \mathcal{O}\left(\frac{1}{c^2}\right) + \mathcal{O}(e_t^3)$$

$$\dot{r} = c\sqrt{x}\left[e_t\sin(l)\left(1 - \frac{3}{8}e_t^2 + \widetilde{\mu}_+^{(2)} x^5\left(-9 - \frac{87}{2}e_t^2\right)\right) + e_t^2\sin(2l)\left(1 - 39\widetilde{\mu}_+^{(2)} x^5\right)\right]$$

$$\dot{\phi} = \frac{x^{3/2}c^3}{GM}\left[1 + e_t\cos(l)\left(2 - \frac{e_t^2}{4} + \widetilde{\mu}_+^{(2)} x^5\left(60 + \frac{1215}{4}e_t^2\right)\right) + \frac{e_t^2}{2}\cos(2l)\left(5 + 180\widetilde{\mu}_+^{(2)} x^5\right)\right]$$

- Additional time scale, $\lambda = Kl$, in the phase $\phi - \phi_0 = \lambda + W(l)$

$$W(l) = e_t\sin(l)\left(2 - \frac{e_t^2}{4} + \widetilde{\mu}_+^{(2)} x^5\left(150 + \frac{1755}{2}e_t^2\right)\right) + \frac{e_t^2}{4}\sin(2l)\left(5 + 405\widetilde{\mu}_+^{(2)} x^5\right) + \mathcal{O}\left(\frac{1}{c^2}\right) + \mathcal{O}(e_t^3)$$

Radiative dynamics : the acceleration at 2.5PN

- Modified geodesic equation [Bailey, Israel 75]

$$\frac{\mathrm{D}p_\mu}{\mathrm{d}\tau} = -\frac{1}{6} J^{\lambda\nu\rho\sigma} \nabla_\mu R_{\lambda\nu\rho\sigma}$$

- (3+1) decomposition

$$\frac{\mathrm{d}P^i}{\mathrm{d}t} = F^i$$

- Insert the PN metric

$$\boldsymbol{a} = -\frac{GM}{r^2} \left(1 + 18 \frac{G\,\mu_+^{(2)}}{r^5}\right) \boldsymbol{n} + \frac{\boldsymbol{a}_{\mathrm{RR}}}{c^5}$$

- Allow time dependency on x, e_t and

$$l(t) = \int_{t_0}^{t} n(\tau)\ \mathrm{d}\tau + c_l(t)$$

$$\lambda(t) = \int_{t_0}^{t} K(\tau)\, n(\tau)\ \mathrm{d}\tau + c_\lambda(t)$$

- Impose condition $\boldsymbol{a}_{\mathrm{RR}}$

$$\frac{\mathrm{d}c_{x,e_t}}{\mathrm{d}t} = \sum_k \frac{\alpha_k}{(1 - e_t \cos u)^k}$$

$$\frac{\mathrm{d}c_{l,\lambda}}{\mathrm{d}t} = \sum_k \frac{\beta_k \sin u + \mu_k (v-u)}{(1 - e_t \cos u)^k}$$

- Split secular and oscillating parts

$$x(t) = \bar{x}(t) + \tilde{x}(t)$$

Results for the oscillatory part (eccentricity expanded)

$$\tilde{x} = \nu x^{7/2}\Bigg\{e_t\sin(l)\left[80+\frac{4538}{15}e_t^2+x^5\left(\frac{\widetilde{\mu}_+^{(2)}+\delta\widetilde{\mu}_-^{(2)}}{\nu}\left(834+\frac{346899}{20}e_t^2\right)+\widetilde{\mu}_+^{(2)}\left(13872+\frac{901252}{5}e_t^2\right)\right)\right]$$
$$+e_t^2\sin(2l)\left[\frac{1436}{15}+x^5\left(\frac{3012}{5}\frac{\widetilde{\mu}_+^{(2)}+\delta\widetilde{\mu}_-^{(2)}}{\nu}+\frac{73324}{5}\widetilde{\mu}_+^{(2)}\right)\right]$$
$$+e_t^3\sin(3l)\left[\frac{6022}{45}+x^5\left(-\frac{52113}{20}\frac{\widetilde{\mu}_+^{(2)}+\delta\widetilde{\mu}_-^{(2)}}{\nu}+\frac{209468}{15}\widetilde{\mu}_+^{(2)}\right)\right]\Bigg\}+O\left(e_t^4\right)$$

$$\tilde{e}_t = -\nu x^{5/2}\Bigg\{\sin(l)\left[\frac{64}{5}+\frac{1138}{15}e_t^2+x^5\left(\frac{\widetilde{\mu}_+^{(2)}+\delta\widetilde{\mu}_-^{(2)}}{\nu}\left(\frac{384}{5}+\frac{75789}{20}e_t^2\right)+\widetilde{\mu}_+^{(2)}\left(\frac{7488}{5}+\frac{168442}{5}e_t^2\right)\right)\right]$$
$$+e_t\sin(2l)\left[\frac{352}{15}+\frac{842}{15}e_t^2+x^5\left(\frac{\widetilde{\mu}_+^{(2)}+\delta\widetilde{\mu}_-^{(2)}}{\nu}\left(\frac{669}{5}+\frac{24164}{5}e_t^2\right)+\widetilde{\mu}_+^{(2)}\left(\frac{16108}{5}+\frac{208598}{5}e_t^2\right)\right)\right]$$
$$+e_t^2\sin(3l)\left[\frac{358}{9}+x^5\left(-\frac{987}{4}\frac{\widetilde{\mu}_+^{(2)}+\delta\widetilde{\mu}_-^{(2)}}{\nu}+\frac{15358}{3}\widetilde{\mu}_+^{(2)}\right)\right]$$
$$+e_t^3\sin(4l)\left[\frac{1289}{20}+x^5\left(-\frac{13999}{5}\frac{\widetilde{\mu}_+^{(2)}+\delta\widetilde{\mu}_-^{(2)}}{\nu}+\frac{22369}{4}\widetilde{\mu}_+^{(2)}\right)\right]\Bigg\}+O\left(e_t^4\right)$$

Outline

Overview of the computations

1) Source multipole moments $\{I_L, J_L\}$ in terms of $(r, \dot{r}, \phi, \dot{\phi})$ [HFB20a, DHB24]

- Compute the stress-energy tensor
- Compute the PN metric $g_{\mu\nu}$ sourced by $T^{\mu\nu}$
- Integrate using Hadamard regularisation (equivalent to dimensional regularisation at 2PN)

2) Orbit averaged fluxes & secular evolution of orbital elements [H26]

- Compute the fluxes (instantaneous and tail)
- Use balance equations to compute the secular evolution of orbital elements

3) Waveform amplitude [H26]

- Instantaneous
- Tail
- Phase redefinition

Radiative multipole moments and fluxes

Fluxes

$$\mathcal{F} = \sum_{\ell=2}^{\infty} \frac{G\,a_\ell}{c^{2\ell+1}} \left[U_L^{(1)} U_L^{(1)} + \frac{4\ell^2}{c^2(\ell+1)^2} V_L^{(1)} V_L^{(1)} \right]$$

$$\mathcal{G}_i = \varepsilon_{iab} \sum_{\ell=2}^{\infty} \frac{G\,b_\ell}{c^{2\ell+1}} \left[U_{aL-1} U_{bL-1}^{(1)} + \frac{4\ell^2}{c^2(\ell+1)^2} V_{aL-1} V_{bL-1}^{(1)} \right]$$

Link between source and radiative moments:

$$U_L(t) = I_L^{(\ell)}(t) + \frac{2G\mathcal{M}}{c^3} \int_0^{+\infty} \mathrm{d}\tau\, I_L^{(\ell+2)}(t-\tau) \ln\left(\frac{\tau}{\tau_\ell}\right) + \mathcal{O}\left(\frac{1}{c^5}\right)$$

- Similar expression for V_L
- $\mathcal{M}$: ADM mass
- Split into **instantaneous** and **tail** parts

Instantaneous fluxes (orbit averaged)

- Compute the instantaneous part in terms of $(r, \dot{r}, \dot{\phi})$

$$\mathcal{F}^{\text{LO}}_{\text{inst}} = \frac{32}{5}\frac{G^3M^4\nu^2}{r^4c^5}\Bigg\{\left(r\dot{\phi}\right)^2 + \frac{\dot{r}^2}{12} + \frac{\mu_+^{(2)} + \delta\,\mu_-^{(2)}}{M\nu r^4}\left[15\dot{r}^4 - \frac{225}{2}\left(\dot{r}r\dot{\phi}\right)^2 + \frac{45}{2}\left(r\dot{\phi}\right)^4 + \frac{GM}{2r}\left(21\dot{r}^2 - 33\left(r\dot{\phi}\right)^2\right)\right] + \frac{12G\mu_+^{(2)}}{r^5}\left(3\left(r\dot{\phi}\right)^2 - \dot{r}^2\right)\Bigg\}$$

- Inject $(r, \dot{r}, \dot{\phi})$ as functions of (x, e_t, u)
- Perform orbit averaging operation

$$\langle A\rangle = \frac{1}{P}\int_0^P \mathrm{d}t\, A = \int_0^{2\pi}\frac{\mathrm{d}u}{2\pi}\frac{\mathrm{d}l}{\mathrm{d}u}A(u)$$

Kernel integrals ($n \geq 1$)

$$\int_0^{2\pi}\frac{\mathrm{d}u}{2\pi}\frac{\cos(ku)}{(1-e\cos u)^n} = \frac{(n+k-1)!}{(n-1)!}\sum_{\ell=0}^{n-1}\frac{1}{2^\ell \ell!(k+\ell)!}\frac{(n+\ell-1)!}{(n-\ell-1)!}\frac{(1-\sqrt{1-e^2})^{k+\ell}}{e^k(1-e^2)^{(n+\ell)/2}}$$

$$\int_0^{2\pi}\frac{\mathrm{d}u}{2\pi}\frac{\ln(1-e\cos u)}{(1-e\cos u)^n} = -\sum_{\ell=0}^{n-1}\left[\ln\left(\frac{1+\sqrt{1-e^2}}{2(1-e^2)}\right) + 2\sum_{i=0}^{\ell-1}\frac{1}{n+i}\right]\frac{1}{2^\ell(\ell!)^2}\frac{(n+\ell-1)!}{(n-\ell-1)!}\frac{(1-\sqrt{1-e^2})^\ell}{(1-e^2)^{(n+\ell)/2}}$$

Resummed tail fluxes

- Select the tail part

$$\mathcal{F}^{\text{LO}}_{\text{tail}} = \frac{4G^2\mathcal{M}}{5c^8} I^{(3)}_{ij}(T_R) \int_0^{\infty} \mathrm{d}\tau \ln\left(\frac{\tau}{\tau_1}\right) I^{(5)}_{ij}(T_R - \tau)$$

- Replace the source moments in terms of (x, e_t, l), integrate and orbit average (set $e^{inl} = 0$)

$$\langle \mathcal{F}^{\text{LO}}_{\text{tail}} \rangle = \frac{128\pi x^{13/2}\nu^2 c^5}{5G} \left\{ 1 + \frac{2335}{192} e_t^2 + 6x^5 \left[\frac{\widetilde{\mu}^{(2)}_+ + \delta\, \widetilde{\mu}^{(2)}_-}{\nu} \left(1 + \frac{7015}{128} e_t^2\right) + \widetilde{\mu}^{(2)}_+ \left(4 + \frac{118255}{384} e_t^2\right) \right] \right\} + \mathcal{O}(e_t^4)$$

- Perform a **resummation** (ansatz on the link between powers of x and $1 - e_t^2$)

$$\begin{aligned} \langle \mathcal{F}^{\text{LO}}_{\text{tail}} \rangle =& \frac{128\pi x^{13/2}\nu^2 c^5}{5G(1-e_t^2)^5} \left\{ 1 + \frac{1375}{192} e_t^2 + \mathcal{O}(e_t^4) \right. \\ & \left. + 6 \frac{x^5}{(1-e_t^2)^5} \left[\frac{\widetilde{\mu}^{(2)}_+ + \delta\, \widetilde{\mu}^{(2)}_-}{\nu} \left(1 + \frac{5735}{128} e_t^2\right) + \widetilde{\mu}^{(2)}_+ \left(4 + \frac{102895}{384} e_t^2\right) + \mathcal{O}(e_t^4) \right] \right\} \end{aligned}$$

- Resummation **not unique**, satisfactory precision for the point mass part (error $\lesssim 10^{-4}$ for all $0 \leq e_t < 1$)

Total energy flux

$$\langle \mathcal{F} \rangle = \frac{32 x^5 c^5 \nu^2}{5G(1-e_t^2)^{7/2}} \left[\mathcal{F}_0 + \frac{x}{1-e_t^2}\mathcal{F}_1 + \frac{\pi\, x^{3/2}}{(1-e_t^2)^{3/2}}\mathcal{F}_{1.5} + \frac{x^2}{(1-e_t^2)^2}\mathcal{F}_2 + \frac{\pi\, x^{5/2}}{(1-e_t^2)^{5/2}}\mathcal{F}_{2.5} \right]$$
$$+ \frac{192 x^{10} c^5 \nu}{5G(1-e_t^2)^{17/2}} \left[\mathcal{F}_5 + \frac{x}{1-e_t^2}\mathcal{F}_6 + \frac{\pi\, x^{3/2}}{(1-e_t^2)^{3/2}}\mathcal{F}_{6.5} + \frac{x^2}{(1-e_t^2)^2}\mathcal{F}_7 + \frac{\pi\, x^{5/2}}{(1-e_t^2)^{5/2}}\mathcal{F}_{7.5} \right]$$

- Long expressions

$$\mathcal{F}_0 = 1 + \frac{73}{24}e_t^2 + \frac{37}{96}e_t^4$$
$$\mathcal{F}_5 = \left(\widetilde{\mu}_+^{(2)} + \delta\, \widetilde{\mu}_-^{(2)}\right) \left[1 + \frac{211}{8}e_t^2 + \frac{3369}{32}e_t^4 + \frac{6275}{64}e_t^6 + \frac{10355}{512}e_t^8 + \frac{225}{512}e_t^{10}\right] + \nu\, \widetilde{\mu}_+^{(2)} \left[-3 + \frac{1247}{12}e_t^2 + \frac{56069}{192}e_t^4 + \frac{5341}{32}e_t^6 + \frac{42019}{3072}e_t^8 + \sqrt{1-e_t^2}\left(7 + \frac{1327}{24}e_t^2 + \frac{1081}{24}e_t^4 + \frac{3335}{384}e_t^6 + \frac{37}{192}e_t^8\right)\right]$$

- Similar expression for the norm of the angular momentum flux $\langle \mathcal{G} \rangle$
- We can deduce the secular evolution of quantities depending on the Noetherian quantities

$$\boxed{\langle \dot{A} \rangle = -\frac{\partial A}{\partial E}\langle \mathcal{F} \rangle - \frac{\partial A}{\partial J}\langle \mathcal{G} \rangle}$$

Secular dynamics : derived to NNLO (displayed at LO)

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} =& \frac{64\,x^5\nu c^3}{5GM(1-e_t^2)^{7/2}}\left(1+\frac{73}{24}e_t^2+\frac{37}{96}e_t^4\right)\\
&+\frac{384\,x^{10}c^3}{5GM(1-e_t^2)^{17/2}}\Bigg\{\left(\widetilde{\mu}_+^{(2)}+\delta\,\widetilde{\mu}_-^{(2)}\right)\left[1+\frac{211}{8}e_t^2+\frac{3369}{32}e_t^4+\frac{6275}{64}e_t^6+\frac{10355}{512}e_t^8+\frac{225}{512}e_t^{10}\right]\\
&\qquad+\nu\,\widetilde{\mu}_+^{(2)}\Bigg[27+\frac{549}{4}e_t^2+\frac{7303}{24}e_t^4+\frac{57727}{384}e_t^6+\frac{39199}{3072}e_t^8\\
&\qquad\quad+\sqrt{1-e_t^2}\left(-5+\frac{1237}{24}e_t^2+\frac{3869}{64}e_t^4+\frac{1813}{192}e_t^6-\frac{29}{128}e_t^8\right)\Bigg]\Bigg\}\\
\frac{\mathrm{d}e_t}{\mathrm{d}t} =& -\frac{304\,x^4\nu c^3 e_t}{15GM(1-e_t^2)^{5/2}}\left(1+\frac{121}{304}e_t^2\right)\\
&-\frac{2448\,x^9c^3e_t}{5GM(1-e_t^2)^{15/2}}\Bigg\{\left(\widetilde{\mu}_+^{(2)}+\delta\,\widetilde{\mu}_-^{(2)}\right)\left[1+\frac{487}{68}e_t^2+\frac{1245}{136}e_t^4+\frac{2545}{1088}e_t^6+\frac{65}{1088}e_t^8\right]\\
&\qquad+\nu\,\widetilde{\mu}_+^{(2)}\Bigg[\frac{479}{102}+\frac{27611}{1224}e_t^2+\frac{96463}{4896}e_t^4+\frac{100463}{39168}e_t^6+\frac{5}{272}e_t^8\\
&\qquad\quad+\sqrt{1-e_t^2}\left(\frac{550}{153}+\frac{7391}{1224}e_t^2+\frac{4057}{4896}e_t^4-\frac{13}{288}e_t^6\right)\Bigg]\Bigg\}
\end{aligned}
$$

Are tidal eccentric corrections detectable? A numerical integration

1) Numerically integrate $x(t)$ and $e_t(t)$

- Initial frequency : 10Hz (LIGO band)
- Final frequency : Schwarzschild ISCO

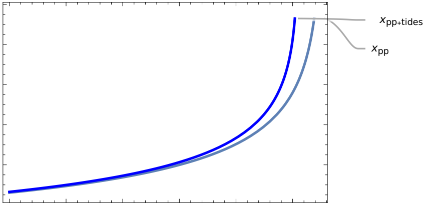
2) Compute t_{ISCO} and deduce the accumulated phase

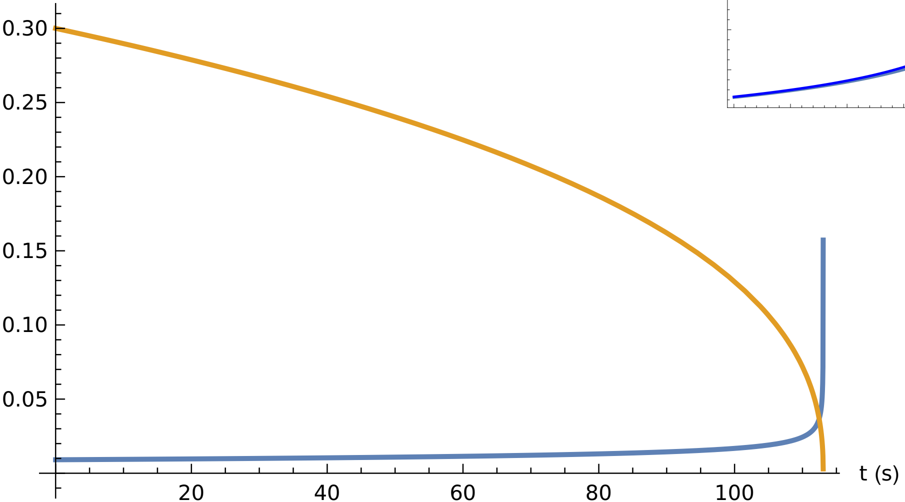
$$\phi_{\text{ISCO}} = \frac{c^3}{GM}\int_0^{t_{\text{ISCO}}} \mathrm{d}t\, x^{3/2}(t)$$

Number of GW cycles: $N_{\text{GW}} = 2N_{\text{orb}} = \phi_{\text{ISCO}}/\pi$

	$m_1(M_\odot)$	$m_2(M_\odot)$	e_0
Case I (NSBH)	11.5	1.5	0.14
Case II (BNS)	1.4	1.4	0.3
Case III (BNS)	1.4	1.4	0.6
Case IV (BNS)	1.8	0.8	0.3

Secular dynamics





Dephasing induced by eccentricity-tidal couplings

$t_{\text{ISCO}}(s)$	QC pp	QC pp & tides	Ecc pp	Ecc pp & tides	$\Delta\phi$	ΔN
Case I	30.1310	30.1309	28.1095	28.1095	0.041	0.013
Case II	160.737	160.734	113.054	113.051	7.54	2.40
Case III	"	"	31.2112	31.2072	7.60	2.42
Case IV	213.083	213.078	149.892	149.887	11.1	3.5

- Detectability threshold depends on SNR ρ : $\Delta\phi \gtrsim \frac{1}{\sqrt{2}\rho}$ [Lindblom, Owen, Brown 2008]
  - GW200105 : effect not detectable ; SNR ~ 13 while threshold ~ 17
  - Other cases : **possibly detectable** for signals with high enough SNR
- Rigorous study required for definite conclusions
  - Compute mismatch between waveforms with and without tides
- Model limitations :
  - Cannot extrapolate equations up to ISCO (strong fields, NSs merge before)
  - Other effects dominate : dynamical tides, tidal disruption, mass transfer, EM fields...
  - Thus, **lower bound on the dephasing** with realistic eccentric BNS

Amplitude modes

- Inject the radiative multipoles in
$$h_{\ell m} = \frac{G\,c_\ell}{Rc^{\ell+2}}\alpha_L^{\ell m}\left(U_L + \frac{2\ell}{\ell+1}\frac{i}{c}V_L\right)$$
- 4 types of contributions : instantaneous, tail, post-adiabatic (RR) and memory
  - Instantaneous, post-adiabatic & tail done
  - Memory left for future work
- Phase redefinition (cancellation of gauge constant in the tails)
$$\xi = l - \frac{3G\mathcal{M}}{c^3} n \ln\left(\frac{x}{x'_0}\right)\,, \qquad \psi = \phi(l \to \xi)$$
- Observable amplitude modes written as
$$h_{\ell m} = \frac{8GM\nu x}{R\,c^2}\sqrt{\frac{\pi}{5}} H^{\psi}_{\ell m}(x, e_t, \xi) e^{-im\psi}$$
- Computations done at relative 2.5PN order, eccentricity expanded to $\mathcal{O}(e_t^{12})$
- **All modes computed** for $2 \leq \ell \leq 7$ and $|m| \leq 7$

The (2,2)-mode at relative 1.5PN and $\mathcal{O}(e_t)$

$$
\begin{aligned}
H^{\psi}_{22} = {}& 1 + e_t\left(\frac{5}{4}e^{i\xi} + \frac{1}{4}e^{-i\xi}\right) + \left[-\frac{107}{42} + \frac{55}{42}\nu + e_t\left(-\frac{31}{24}e^{i\xi} - \frac{257}{168}e^{-i\xi} + \nu\left(\frac{35}{24}e^{i\xi} + \frac{169}{168}e^{-i\xi}\right)\right)\right]x \\
& + \left[2\pi + \frac{e_t}{4}\left(e^{i\xi}\left(13\pi + 6i\ln 2\right) + e^{-i\xi}\left(11\pi + 54i\ln(3/2)\right)\right)\right]x^{3/2} \\
& + \left\{\frac{\widetilde{\mu}^{(2)}_{+} + \delta\,\widetilde{\mu}^{(2)}_{-}}{\nu}\left[3 + \frac{e_t}{8}\left(45e^{i\xi} + 141e^{-i\xi}\right)\right] + \widetilde{\mu}^{(2)}_{+}\left[12 + \frac{e_t}{2}\left(105e^{i\xi} + 111e^{-i\xi}\right)\right]\right\}x^{5} \\
& + \left\{\frac{\widetilde{\mu}^{(2)}_{+} + \delta\,\widetilde{\mu}^{(2)}_{-}}{\nu}\left[\frac{9}{2} + \frac{125}{7}\nu + e_t\left(33e^{i\xi} + \frac{111}{2}e^{-i\xi} + \nu\left(\frac{12573}{112}e^{i\xi} + \frac{11369}{112}e^{-i\xi}\right)\right)\right]\right. \\
& \left. - \widetilde{\mu}^{(2)}_{+}\left[\frac{265}{7} - \frac{45}{7}\nu - e_t\left(\frac{263}{28}e^{i\xi} + e^{-i\xi}\left(\frac{3683}{28} - \frac{153}{14}\nu\right)\right)\right] + \widetilde{\sigma}^{(2)}_{+}\left[\frac{224}{3} + \frac{e_t}{3}\left(1574e^{i\xi} + 1262e^{-i\xi}\right)\right]\right\}x^{6} \\
& + \left\{\frac{\widetilde{\mu}^{(2)}_{+} + \delta\,\widetilde{\mu}^{(2)}_{-}}{\nu}\left[6\pi + \frac{3e_t}{8}\left(e^{i\xi}\Big(31\pi + 2i\ln(2)\Big) + e^{-i\xi}\Big(157\pi + 378i\ln(3/2)\Big)\right)\right]\right. \\
& \left. + \widetilde{\mu}^{(2)}_{+}\left[24\pi + \frac{3e_t}{4}\left(e^{i\xi}\Big(257\pi + 18i(23\ln(2) - 5)\Big) + e^{-i\xi}\Big(319\pi + 54i(29\ln(3/2) - 5)\Big)\right)\right]\right\}x^{13/2}
\end{aligned}
$$

Outline

Summary

- GW science is dependent on **template accuracy**
- Important to **include physical effects** in waveform models
- **PN is essential** for waveform modeling
- Results : Adiabatic tides on **quasi-circular orbits** to relative 2.5PN
  - Complete dynamics & waveform, ready-to-use results available in ancillary file of [DHB24]
  - Implemented in all waveform models describing tidal effects
- Results : Adiabatic tides on **eccentric orbits** to relative 2.5PN
  - Dynamics fully solved
  - Waveform almost complete $\rightarrow$ missing memory effects
  - Results available in ancillary files of [HH25, H26]

Perspectives:

- Complete memory contributions
- Consider dynamical tides
  - Quasi-circular orbits
  - Eccentric orbits

Acknowledgments

Thank you !

This work was supported by the Universitat de les Illes Balears (UIB); the Spanish Agencia Estatal de Investigación grants PID2022-138626NB-I00, RED2022-134204-E, RED2022-134411-T, funded by MICIU/AEI/10.13039/501100011033 and the ERDF/EU; and the Comunitat Autònoma de les Illes Balears through the Conselleria d'Educació i Universitats with funds from the European Union - NextGenerationEU/PRTR-C17.I1 (SINCO2022/6719) and from the European Union - European Regional Development Fund (ERDF) (SINCO2022/18146).