Spin-precessing gravitational waveforms: an analytic perspective

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Paris, Jan. 11 2016

Outline

1. Gravitational Waves
2. Precessing Binaries
3. Radiation reaction
4. Waveform building
5. Conclusion

Gravitational Waves

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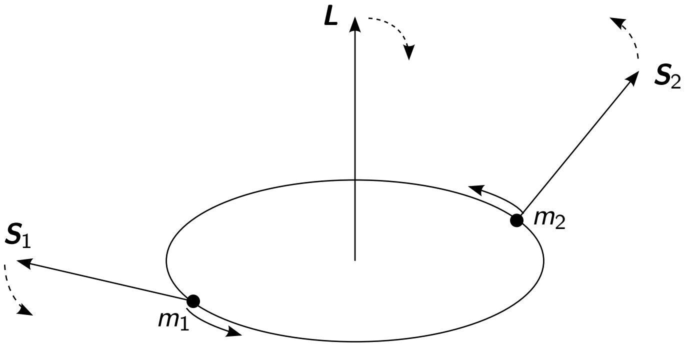
Spin-induced precession breaks degeneracies: greatly improves parameter estimation.

Spin-Orbit Precession

Binaries of spinning objects undergo precession



Equations of motion

PN parameter

$$v = (M\omega)^{1/3}$$

$$\dot{v} = v^9 \sum_{n \geq 0} a_n v^n \quad \Longleftrightarrow \quad T_{rr} = \mathcal{O}(v^{-8})$$

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$$\left.\begin{aligned} \dot{\hat{\boldsymbol{L}}} &= v^6 \left(A_1 \boldsymbol{S}_1 + A_2 \boldsymbol{S}_2\right) \times \hat{\boldsymbol{L}} \\ \dot{\boldsymbol{S}}_1 &= v^5 A_1 \hat{\boldsymbol{L}} \times \boldsymbol{S}_1 + v^6 A_{12} \boldsymbol{S}_2 \times \boldsymbol{S}_1 \\ \dot{\boldsymbol{S}}_2 &= v^5 A_2 \hat{\boldsymbol{L}} \times \boldsymbol{S}_2 + v^6 A_{12} \boldsymbol{S}_1 \times \boldsymbol{S}_2 \end{aligned}\right\} \quad \Longleftrightarrow \quad T_{prec} = \mathcal{O}(v^{-5})$$

Solution in the absence of radiation reaction

Equations of precession can be solved analytically, in absence of radiation reaction, in orbit-averaged form, at leading post-Newtonian order [Kesden et al., Phys. Rev. Lett. 114, 081103 (2015)].

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- Second step: identify suitable parametrization.

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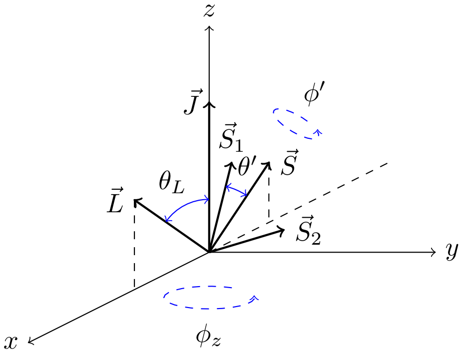
Two dynamical quantities.

Choice of frames



Choice of frames: $\hat{\boldsymbol{z}} = \hat{\boldsymbol{J}}$, $\hat{\boldsymbol{z}}' = \hat{\boldsymbol{S}}$.

Choice of frames

Because of conserved quantities, we can select two parameters to describe the evolution: ϕ_z and S.
E.g. $\boldsymbol{S} = \boldsymbol{J} - \boldsymbol{L} \quad \Longrightarrow \quad J^2 + L^2 - 2JL\cos\theta_L = S^2$.

Evolution of S

Equations of motion:

$$\left(\frac{dS^2}{dt}\right)^2 = -A^2\left(S^2 - S_+^2\right)\left(S^2 - S_-^2\right)\left(S^2 - S_3^2\right).$$

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Solution:

$$S^2 = S_+^2 + (S_-^2 - S_+^2)\,\mathrm{sn}^2(\psi, \mathrm{m}),$$
$$\dot{\psi} = \frac{A}{2}\sqrt{S_+^2 - S_3^2},$$
$$m = \frac{S_+^2 - S_-^2}{S_+^2 - S_3^2}.$$

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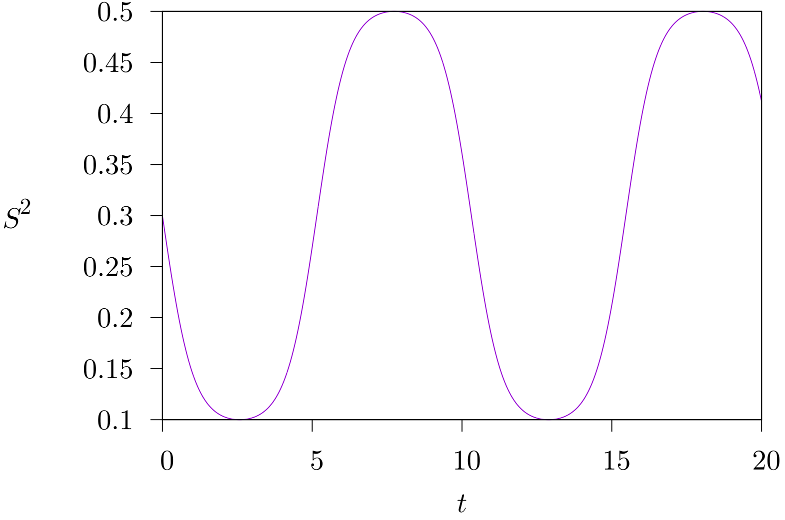

Evolution of ϕ_z

Equations of motion:

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Analytic solution: a complicated combination of elliptic integrals.

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- No radiation reaction.

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- Solve equations order by order.

Constants

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To be able to use the solution previously found, we need to use $\langle \boldsymbol{J} \rangle_{prec}$ to describe the solution, so that $\boldsymbol{J}(t_{rr})$.

Solution for L

Solution for L is very simple: $L = \mu/v$

$$\dot{v} = v^9 \sum_{n \geq 0} a_n v^n$$

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$v \ll 1 \quad \Longrightarrow \quad$ solve order by order.

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$$\left\langle \frac{dJ}{dL} \right\rangle_{prec} = \left\langle \hat{\boldsymbol{J}} \cdot \hat{\boldsymbol{L}} \right\rangle_{prec} = \frac{1}{2JL} \left(J^2 + L^2 - \left\langle S^2 \right\rangle_{prec} \right).$$

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Solution

$$J^2(L) = L^2 + CL - L \int \frac{\left\langle S^2 \right\rangle_{prec}}{L^2} dL.$$

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$$\times [S^2(t_{rr}, t_{prec}) - S_-^2(t_{rr})][S^2(t_{rr}, t_{prec}) - S_3^2(t_{rr})].$$

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$$\left(\frac{\partial S_0^2}{\partial t_{prec}}\right)^2 = -A^2(t_{rr})[S_0^2(t_{rr}, t_{prec}) - S_+^2(t_{rr})] \times [S_0^2(t_{rr}, t_{prec}) - S_-^2(t_{rr})][S_0^2(t_{rr}, t_{prec}) - S_3^2(t_{rr})].$$

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We know solution

$$S_0^2 = S_+^2(t_{rr}) + \left[S_-^2(t_{rr}) - S_+^2(t_{rr})\right] sn[\psi(t_{rr}, t_{prec}), m(t_{rr})],$$

with

$$\dot{\psi} = \frac{A(rr)}{2}\sqrt{S_+^2(t_{rr}) - S_3^2(t_{rr})}.$$

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Same treatment, except:

$$\sum_{n\geq -1} \left(\epsilon^n \frac{\partial \phi_z^{(n)}}{\partial t_{prec}} + \epsilon^{n+1} \frac{\partial \phi_z^{(n)}}{\partial t_{rr}} \right) = \Omega_z[S(t_{rr}, t_{prec}), L(t_{rr}), J(t_{rr})].$$

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$\mathcal{O}(\epsilon^{-1})$:

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$\phi_z^{(-1)} = \phi_z^{(-1)}(t_{rr})$.

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$\mathcal{O}(\epsilon^0)$:

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$\mathcal{O}(\epsilon^0)$:

$$\frac{\partial \phi_z^{(0)}}{\partial t_{prec}} + \frac{\partial \phi_z^{(-1)}}{\partial t_{rr}} = \Omega_z^{(0)}(t_{prec}, t_{rr}).$$

Averaging over T_{prec}:

$$\frac{d\phi_z^{(-1)}}{dt_{rr}} + \left\langle \frac{\partial \phi_z^{(0)}}{\partial t_{prec}} \right\rangle_{prec} = \left\langle \Omega_z^{(0)} \right\rangle_{prec}$$

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$$\frac{d\phi_z^{(-1)}}{dt_{rr}} = \left\langle \Omega_z^{(0)} \right\rangle_{prec} (t_{rr}).$$

Regular post-Newtonian integration.

Solution for ϕ_z

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We can use the previous solution, provided we subtract the precession average.

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- Radiation reaction present.

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The signal observed by a detector can be expressed by

$$h(t) = Re\left[(F_+ + iF_\times)(h_+ - ih_\times)\right]$$

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Waveform decomposed in spin-weighted spherical harmonic basis

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Solution for ϕ_{orb} similar to solution for L.

Waveform

$$H^{lm} = h^{lm}(\iota) \sum_{m'=-l}^{l} D^{l}_{m',m}(\theta_L, \phi_z, \zeta)\,{}_{-2}Y_{lm'}(\theta_s, \phi_s),$$

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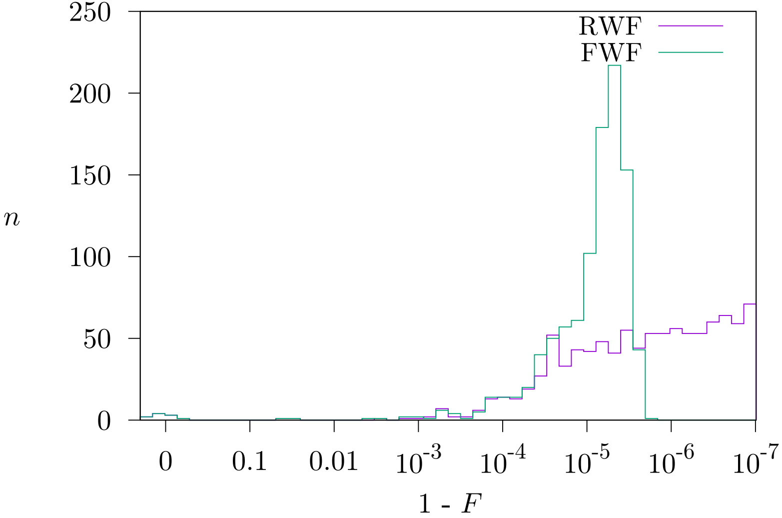
Solution for ζ similar to solution for ϕ_z.

Waveform

To compute the Fourier transform, use SUA:

$$\tilde{h}(f) = \sqrt{2\pi} \sum_{m \geq 1} T_m e^{2\pi i f t_m - m\Phi - \pi/4} \times \sum_{l \geq 2} \sum_{k=-k_{\max}}^{k_{\max}} \frac{a_{k,k_{\max}}}{2 - \delta_{k,0}} \mathcal{H}_{lm}(t_m + kT_m)$$

Comparisons: neutron star-neutron star



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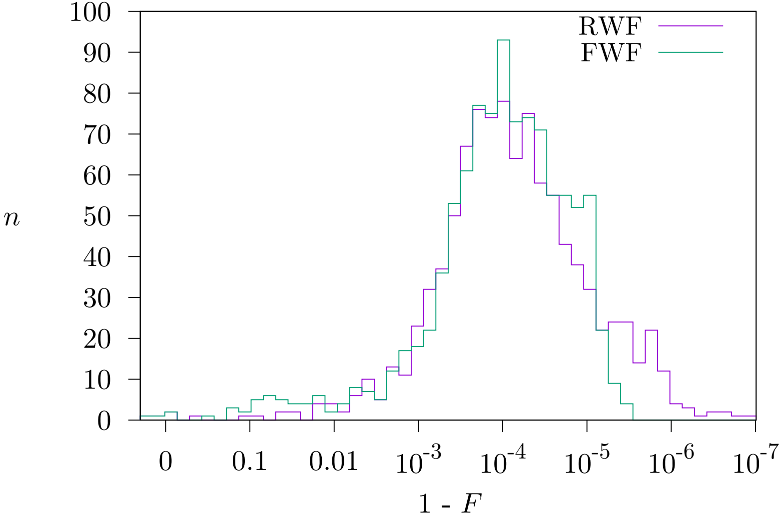

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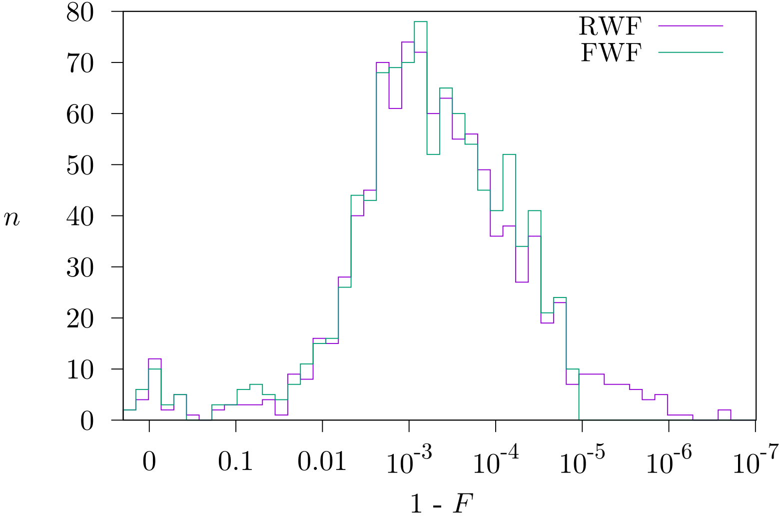

Last problem: ϕ_z and ζ

Equation of motion for ϕ_z:

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When a root of the denominator polynomial is small, we run into problems.

Solution still to be found.

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- More accurate precession: next-to-leading order spin-spin terms? Conserved quantity ξ still present?

Thank you!