Black hole perturbations in modified gravity

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May 10, 2021

GRεCO seminar - Institut d’Astrophysique de Paris
Introduction

- Modified gravity theories: predictions different from GR
- Important test: quasinormal modes of black holes
- Up to now, theoretical computations are rare
- Present a systematic algorithm to extract physical information and perform numerical analysis
Outline

1. Modified gravity: DHOST theories
   - Necessity for modified gravity
   - Importance of black holes

2. Quasinormal modes in GR
   - Perturbation setup
   - Schrödinger equations

3. Quasinormal modes in modified gravity
   - Similarities and differences
   - QNFs from the first order system
   - Numerical results
Modified gravity: DHOST theories
Motivation for beyond-GR theories

Testing deviations

- Design new tests of GR
- Know where to look in large amounts of data

Issues of GR

- Big Bang singularity
- Black hole interior singularity
- Dark energy

⇒ Important to look for extensions of GR
⇒ Recent and near-future experiments will give much insight
Various theories of modified gravity

Lovelock’s theorem for gravity

- Fourth dimensional spacetime
- Only field is the metric
- Second order derivatives in equations

⇒ GR is the only possible theory

General procedure to construct a modified gravity theory:

Break one of Lovelock’s hypotheses → Make sure the theory is not pathological → Take experimental constraints into account
Degenerate Higher-Order Scalar-Tensor theories

DHOST: add a scalar field and higher derivatives

DHOST action

Ingredients: metric $g_{\mu\nu}$, scalar field $\phi$ of kinetic energy $X = \phi_\mu \phi^\mu$ with $\phi_\mu = \nabla_\mu \phi$.

\[
S[g_{\mu\nu}, \phi] = \int d^4x \sqrt{-g} \left( F(X) R + P(X) + Q(X) \Box X + A_1(X) \phi_{\mu\nu} \phi^{\mu\nu} + A_2(X) (\Box \phi)^2 + A_3(X) \phi^\mu \phi_{\mu\nu} \phi^\nu \Box \phi + A_4(X) \phi^\mu \phi_{\mu\nu} \phi^\nu \phi_\rho \phi_\rho + A_5(X) (\phi^\mu \phi_{\mu\nu} \phi^\nu)^2 \right)
\]

Degeneracy and stability: $A_2$, $A_4$ and $A_5$ are not free functions

$\Rightarrow$ Most general scalar-tensor theory

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## Horndeski theory

Simplify the theory:

\[
\tilde{g}_{\mu \nu} = A(X)g_{\mu \nu} + B(X)\phi_\mu \phi_\nu
\]

### DHOST theory
- Higher derivatives
- 5 free functions

### Horndeski theory
- Second-order derivatives only
- 3 free functions

\[
S[g_{\mu \nu}, \phi] = \int d^4x \left( F(X)R + P(X) + Q(X)\Box X + 2F'(X) \left( \phi_\mu \phi^{\mu \nu} - (\Box \phi)^2 \right) \right)
\]

\[\Rightarrow \text{In the following, consider Horndeski}\]

In vacuum both theories are equivalent but the solutions may differ\(^2\).

Tests of modified gravity

Where to look for traces of modified gravity?

- **Black holes**
  - New solutions
  - Different dynamics

- **Large scale structures**
  - Different growth rate
  - Screenings

- **Cosmology**
  - Primordial GWs
  - CMB

- Each theory is tuned for a specific energy scale
- We focus on modifications of gravity in the black hole regime
Quasinormal modes and the ringdown

Ringdown of a merger: excited BH emits GW at precise frequencies, the quasinormal modes

Figure 1: Ringdown phase of a binary black hole merger (L. London 2017)
Measuring quasinormal modes

- Discrete set (similar to plucked string)
- Complex frequencies: energy loss due to emission towards infinity
- Depend a lot of the theory → very good test

Figure 2: Principle of ringdown fit\textsuperscript{3} and application to GW150914\textsuperscript{4}.

\textsuperscript{3} Kokkotas, K. D. and Schmidt, B. G. 1999.
New black holes in DHOST: stealth solution

Metric sector: mimic GR

\[ ds^2 = -(1 - \mu/r) \, dt^2 + (1 - \mu/r)^{-1} \, dr^2 + r^2 \, d\Omega^2 \]

Scalar sector

\[ \phi = qt + \psi(r) \]
\[ X = -q^2 \Rightarrow \psi'(r) = q \frac{\sqrt{r\mu}}{r - \mu} \]

- Metric sector: similar to Schwarzschild ⇒ existing background tests still valid
- Scalar sector: time-dependant field and constant kinetic term
- Parametrization on \( F, P \) and \( Q \) for existence:

\[ F(X) = 1, \quad F'(X) = \alpha, \quad F''(X) = \beta \]
\[ P(X) = 0, \quad P'(X) = 0, \quad P''(X) = \gamma \]
\[ Q(X) = 0, \quad Q'(X) = 0, \quad Q''(X) = \delta \]
New black holes in DHOST: BCL solution

Parameters of Horndeski:

\[ F(X) = f_0 + f_1 \sqrt{X} \quad P(X) = -p_1 X, \quad Q(X) = 0 \]

Metric sector: RN with imaginary charge

\[ ds^2 = -A(r) \, dt^2 + \frac{1}{A(r)} \, dr^2 + r^2 \, d\Omega^2 \]
\[ A(r) = 1 - \frac{r_m}{r} - \xi \frac{r_m^2}{r^2} , \quad \xi = \frac{f_1^2}{2f_0 p_1 r_m^2} \]

Scalar sector

\[ \phi = \psi(r) , \quad \psi'(r) = \pm \frac{f_1}{p_1 r^2 \sqrt{A(r)}} \]
\[ X(r) = \frac{f_1^2}{p_1^2 r^4} \]

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Quasinormal modes in GR
Perturbation setup

Schrödinger equations

Separating the degrees of freedom

1. Start with the Einstein-Hilbert action

\[ S[g_{\mu\nu}] = \int d^4x \sqrt{-g} \, R \]

2. Static spherically symmetric background

\[ \tilde{g}_{\mu\nu} = \text{diag}(-A(r), 1/A(r), r^2, r^2 \sin^2 \theta), \quad A(r) = 1 - r_s/r \]

3. Perturb the metric: \( g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu} \) and linearise Einstein’s equations

\[ \Rightarrow \text{obtain 10 equations} \]

4. Decompose the components of \( h_{\mu\nu} \) over spherical harmonics

5. Separate by parity: polar (even) and axial (odd) modes

6. Gauge fixing via \( h_{\mu\nu} \rightarrow h_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \):
   - Polar modes: 7 equations for \( K, H_0, H_1, H_2 \)
   - Axial modes: 3 equations for \( h_0, h_1 \)

7. Fourier transform:

\[ f(t, r) = \exp(-i\omega t)f(r) \]
Reducing the number of equations

Two systems with more equations than variables → overconstrained?

Axial modes
• 2 first-order equations
• 1 second-order equation

Polar modes
• 4 first-order equations
• 2 second-order equations
• 1 algebraic equation

Interestingly, each system is equivalent to a 2-dimensional system of the form

\[
\frac{dX}{dr} = M(r)X
\]

Final system of equations

Axial modes

\[ X_{\text{axial}} = t \begin{pmatrix} h_0 & h_1/\omega \end{pmatrix} \]

\[ M_{\text{axial}} = \begin{pmatrix} \frac{2}{r} & 2i\lambda \frac{r-r_s}{r^3} - i\omega^2 \\ -\frac{r^2}{(r-r_s)^2} & -\frac{r_s}{r(r-r_s)} \end{pmatrix} \]

(set \(2(\lambda + 1) = \ell(\ell + 1)\))

Polar modes

\[ X_{\text{polar}} = t \begin{pmatrix} K & H_1/\omega \end{pmatrix} \]

\[ M_{\text{polar}} = \frac{1}{3r_s + 2\lambda r} \begin{pmatrix} a_{11}(r) + b_{11}(r)\omega^2 & a_{12}(r) + b_{12}(r)\omega^2 \\ a_{21}(r) + b_{21}(r)\omega^2 & a_{22}(r) + b_{22}(r)\omega^2 \\ \frac{r(r-r_s)}{2(r-r_s)^2} & \frac{r^2}{r(r-r_s)} \end{pmatrix} \]

⇒ goal to achieve: simplify these complicated differential systems
Effect of a change of variables

Changing the functions in $X$ is not a change of basis for $M$!

**Change of variables**

\[
\frac{dX}{dr} = M(r)X, \quad X = P(r)\tilde{X}
\]

\[
\frac{d\tilde{X}}{dr} = \tilde{M}(r)\tilde{X}, \quad \tilde{M} = P^{-1}MP - P^{-1}\frac{dP}{dr}
\]

**Main idea:** find a change of variables that will put the equation into a better form
Usual change of variables: propagation equation

Canonical form for $\tilde{M}$:

$$\tilde{M} = \begin{pmatrix} 0 & 1 \\ V(r) - \frac{\omega^2}{c^2} & 0 \end{pmatrix}$$

Physical interpretation

$$\begin{cases} \dot{\tilde{X}}_0 = \tilde{X}_1 \\ \dot{\tilde{X}}_1 = (V(r) - \frac{\omega^2}{c^2})\tilde{X}_0 \end{cases} \Rightarrow \frac{d^2\tilde{X}_0}{dr_*^2} + \left(\frac{\omega^2}{c^2} - V(r)\right)\tilde{X}_0 = 0, \quad \frac{dr}{dr_*} = A(r)$$

Schrödinger equation with potential $V$

$r_*$: “tortoise coordinate”, $r = r_s \rightarrow r_* = -\infty$ and $r = +\infty \rightarrow r_* = +\infty$
Interpretation of the equations

Axial case:

\[ P_{\text{axial}} = \begin{pmatrix} 1 - r_s/r & r \\ ir^2/(r - r_s) & 0 \end{pmatrix}, \quad c = 1 \]

At the horizon and infinity:

\[ X_0(t, r) \propto e^{-i\omega(t\pm r_s)} \]

\[ \Rightarrow \text{Propagation of waves} \]

Physical interpretation

- Free propagation at \( c = 1 \) near the horizon and infinity
- Scattering by the potential \( V \)
- At infinity: recover gravitational waves in Minkowski
Computation of the modes

Quasinormal modes

- Waves *ingoing* at the horizon: $e^{-i\omega(t+r_*)}$
- Waves *outgoing* at infinity: $e^{-i\omega(t-r_*)}$

- 2 boundary conditions + 2nd order system $\rightarrow$ conditions on $\omega$
- “Eigenvalue problem”: find values of parameter such that solutions exist
- Very different from plucked string: wave propagation at each boundary!
Quasinormal modes in modified gravity
Summary: computation of QNMs in GR

1. Linearized Einstein's eqs
2. Gauge choice
3. Background
4. First-order system
5. Schrödinger equations
6. Numerical computation

Major difficulties:
1. Many different theories
3. Many different backgrounds
5. Highly non-trivial change of variables!
New challenges in modified gravity

New theories

**Scalar-tensor**: new scalar degree of freedom that couples to the polar mode

New backgrounds

**Stealth solution**: time-depandaent scalar field, lose staticity

Schrödinger equation

In general, very hard to solve:

\[
\begin{pmatrix}
0 & 1 \\
V(r) - \frac{\omega^2}{c^2} & 0
\end{pmatrix} = P^{-1}MP - P^{-1}\frac{dP}{dr}
\]

⇒ need for a systematic approach that does not rely on specific simplifications
Example: polar BCL perturbations

\[ A(r) = 1 - \frac{r_m}{r} - \zeta \frac{r_m^2}{r^2}, \quad \zeta = \frac{f_1^2}{2f_0p_1r_m^2}, \quad \phi'(r) = \pm \frac{f_1}{p_1 r^2 \sqrt{A(r)}} \]

\[
M(r) = \begin{pmatrix}
\frac{\omega^2 r^2}{A^2} & -\frac{1}{r} + \frac{U}{2r^3 A} & \frac{U}{r^4} & -\frac{i(1+\lambda)}{\omega r^2} & -\frac{\lambda}{A} & -\frac{V}{r^3} \\
-\frac{\lambda}{A} & -\frac{r_m}{2r A} + \frac{r_m^2 S}{4r^4 A^2} & -\frac{2}{r} - \frac{2i\omega}{r} + \frac{2r^5 A}{2i\omega U} & \frac{i(1+\lambda)U}{2r^3 \omega A} & -\frac{3U}{2r^3 A} & -\frac{\zeta^2 r_m^4}{2r^4 A} \\
-\frac{1}{r} + \frac{U}{2r^3 A} & \frac{r^2 A}{r^2} & -\frac{U}{r^4} & -\frac{i\omega}{r^3 A} & -\frac{i(1+\lambda)U}{2r^6 A} & -\frac{V}{r^3} \\
-\frac{1}{r} + \frac{U}{2r^3 A} & \frac{r^2 A}{r^2} & -\frac{U}{r^4} & -\frac{i\omega}{r^3 A} & -\frac{i(1+\lambda)U}{2r^6 A} & -\frac{V}{r^3} \\
\end{pmatrix}
\]

\[ U(r) = r_m(r + \zeta r_m), \quad V(r) = r^2 + \zeta r_m^2, \quad S(r) = r^2 + 2\zeta r(2r_m - r) + 2\zeta^2 r_m^2. \]
First-order system and boundary conditions

Main idea

Skip step 5: get boundary conditions and perform numerical computations from the first-order system

Steps to perform

• Find asymptotic behaviour at the horizon and infinity
• Identify ingoing and outgoing modes
• Use a numerical method that does not require Schrödinger equations

Naively:

\[
\frac{dX}{dr} = MX, \quad M(r) = M_p r^p + O(r^{p-1}) \quad \Rightarrow \quad X \sim \exp \left( \frac{M_p r^{p+1}}{p + 1} \right) X_c
\]
Failure of naive approach

Axial Schwarzschild

\[ M(r) = \begin{pmatrix} 0 & -i\omega^2 \\ -i & 0 \end{pmatrix} + O\left(\frac{1}{r}\right) \]

\[ X \sim \begin{pmatrix} e^{i\omega r} & 0 \\ 0 & e^{-i\omega r} \end{pmatrix} X_c \]

Polar Schwarzschild

\[ M(r) = \begin{pmatrix} 0 & 0 \\ \frac{i\omega^2}{\lambda} & 0 \end{pmatrix} r^2 + O(r) \]

\[ X \sim \begin{pmatrix} 1 & 0 \\ \frac{i\omega^2}{\lambda} \frac{r^3}{3} & 1 \end{pmatrix} X_c \]

**Problem**

- We do not recover the \( e^{\pm i\omega r} \) behaviour all the time!
- This is because of a *nilpotent* leading order in the polar case
- A more advanced mathematical treatment is needed
Mathematical results

Solution: behaviour studied in\(^7\), mathematical algorithm from\(^8\)

**Mathematical algorithm**

Main idea: _diagonalize \( M \) order by order_ using

\[
\tilde{M} = P^{-1}MP - P^{-1}\frac{dP}{dr}
\]

\( \Rightarrow \) important result: diagonalization is _always possible!_

General result:

\[
\begin{align*}
M &= M_p r^p + M_{p-1} r^{p-1} + \ldots \\
\tilde{M} &= D_q r^q + D_{q-1} r^{q-1} + \ldots \\
X &\sim e^{D(r)} r^{D-1} F(r) X_c
\end{align*}
\]

\(^7\) Wasow, W. 1965.

\(^8\) Balser, W. 1999.
Example for the BCL solution: polar perturbations

\[ \tilde{\mathcal{M}} \sim \begin{pmatrix} -i\omega/c_0 & i\omega/c_0 \\ i\omega/c_0 & 1/2 \\ 1/2 & 1 \end{pmatrix} \]

Horizon

\[ \tilde{\mathcal{M}} \sim \begin{pmatrix} -i\omega & i\omega \\ i\omega & -\sqrt{2}\omega \\ \sqrt{2}\omega & \sqrt{2}\omega \end{pmatrix} \]

Infinity

Gravitational part

Scalar part

What can we deduce from this?

- We decoupled both modes but only \textit{locally}
- The gravitational mode propagates at \( c = 1 \) at infinity and \( c_0 \) at the horizon
- Always one ingoing and one outgoing gravitational mode
- The scalar mode does not propagate
“Recipe” for the computation of quasinormal modes

1. Linearized Einstein’s eqs →
2. Gauge choice →
3. Background →
4. First-order system →
5. Asymptotical behaviour →
6. Numerical computation

- Generic algorithm that should work for any modified gravity theory
- Go around the technical difficulties of steps 1 and 3
- Caveat: we do not get the full decoupled equations for the modes ⇒ impossible to get a potential
- Asymptotical behaviour is enough to obtain boundary conditions for numerical resolution
Numerical method

Decomposition onto Chebyshev polynomials $T_n$: $f = \sum_{i=0}^{N} f_i T_i$

**ODE**

$$X = \begin{pmatrix} X_0 & \ldots & X_n \end{pmatrix}^t$$

$$\frac{dX}{dr} = M(r, \omega)X$$

+ boundary conditions

**Numerical system**

$$X = \begin{pmatrix} X_{0i} & \ldots & X_{ni} \end{pmatrix}^t$$

$$D_{ij}X_j = M_{ij}(\omega)X_j$$

+ boundary conditions

- Linear algebra problem: generalized eigenvalue problem
- Procedure: find $\omega$ for $N = N_0$, then $N = N_1 > N_0$, keep the common values
BCL axial modes

Figure 3: Axial QNMs found for the BCL solution with $\xi = 0.5, r_m = 1, \lambda = 2$. 
**BCL polar modes**

- Impose gravitational mode b.c. at horizon and infinity
- Obtain modes even though the full system is not decoupled!

![Graph](image)

**Figure 4:** Polar gravitational QNMs found for the BCL solution with $\zeta = 10^{-4}$, $r_m = 1$, $\lambda = 2$. 

- $N = 25$
- $N = 50$
- Detected modes
Isospectrality

Figure 5: Tracking of the fundamental mode for axial and polar gravitational modes as $\xi$ varies.
Conclusion

- Computing quasinormal modes can be very difficult in modified theories of gravity.
- We propose a new technique: use the first-order system instead of looking for Schrödinger-like equations.
- A mathematical algorithm enables us to decouple the modes asymptotically, which allows us to find their physical behaviour and obtain boundary conditions.
- We can use these boundary conditions to numerically compute the quasinormal modes frequencies.
- The method is theory-agnostic: it can be applied to any theory of gravity and any background.
Thank you for your attention!