Singular Perturbations and Gravitational Lensing

Singular perturbation example: $\epsilon x^2 + x + 1 = 0$

Unperturbed solution: $\epsilon = 0 \quad \rightarrow \quad x = -1$

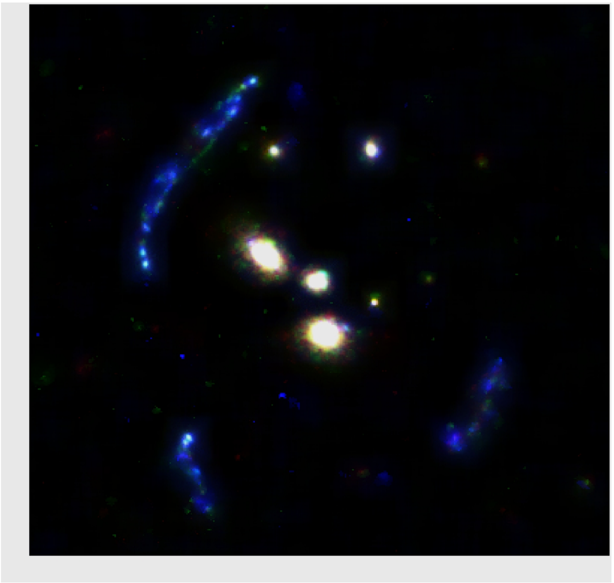
Perturbed solution: $\|\epsilon\| > 0 \quad \rightarrow$ 2 roots

One of the root goes to infinity when $\epsilon \rightarrow 0$

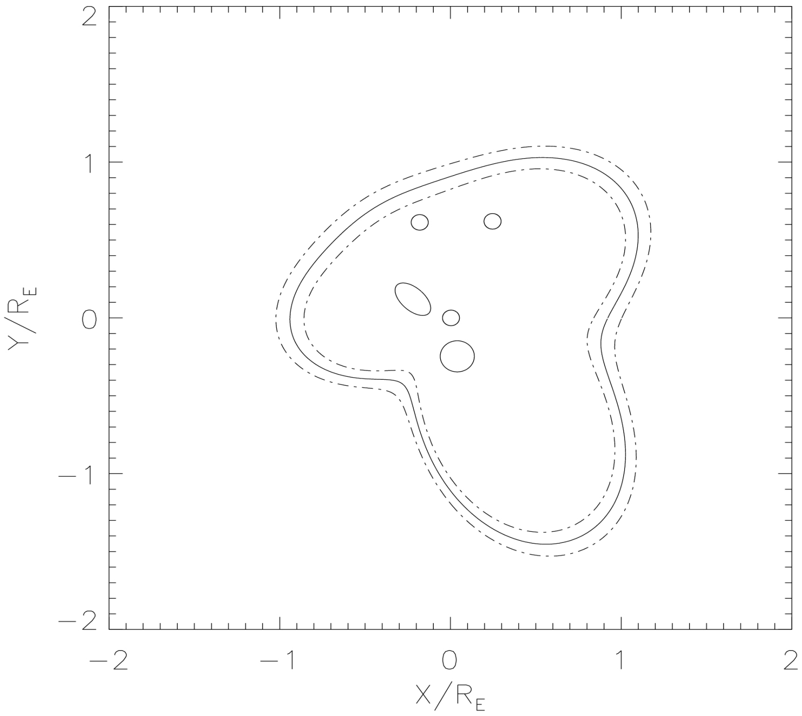
Perturbation is singular

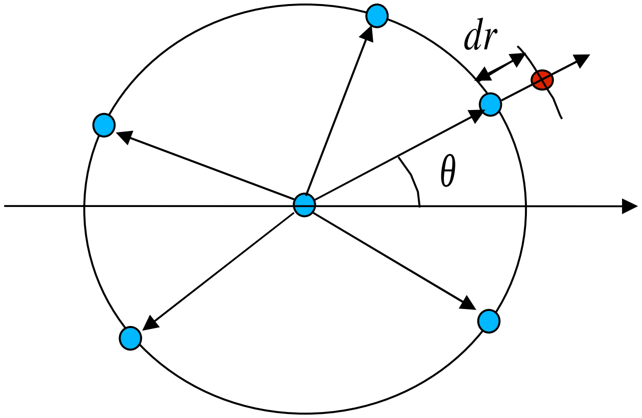
Are gravitational lenses related to singular perturbation theory ?

Singular perturbative theory of strong lenses
A new approach to map dark matter at galactic scale



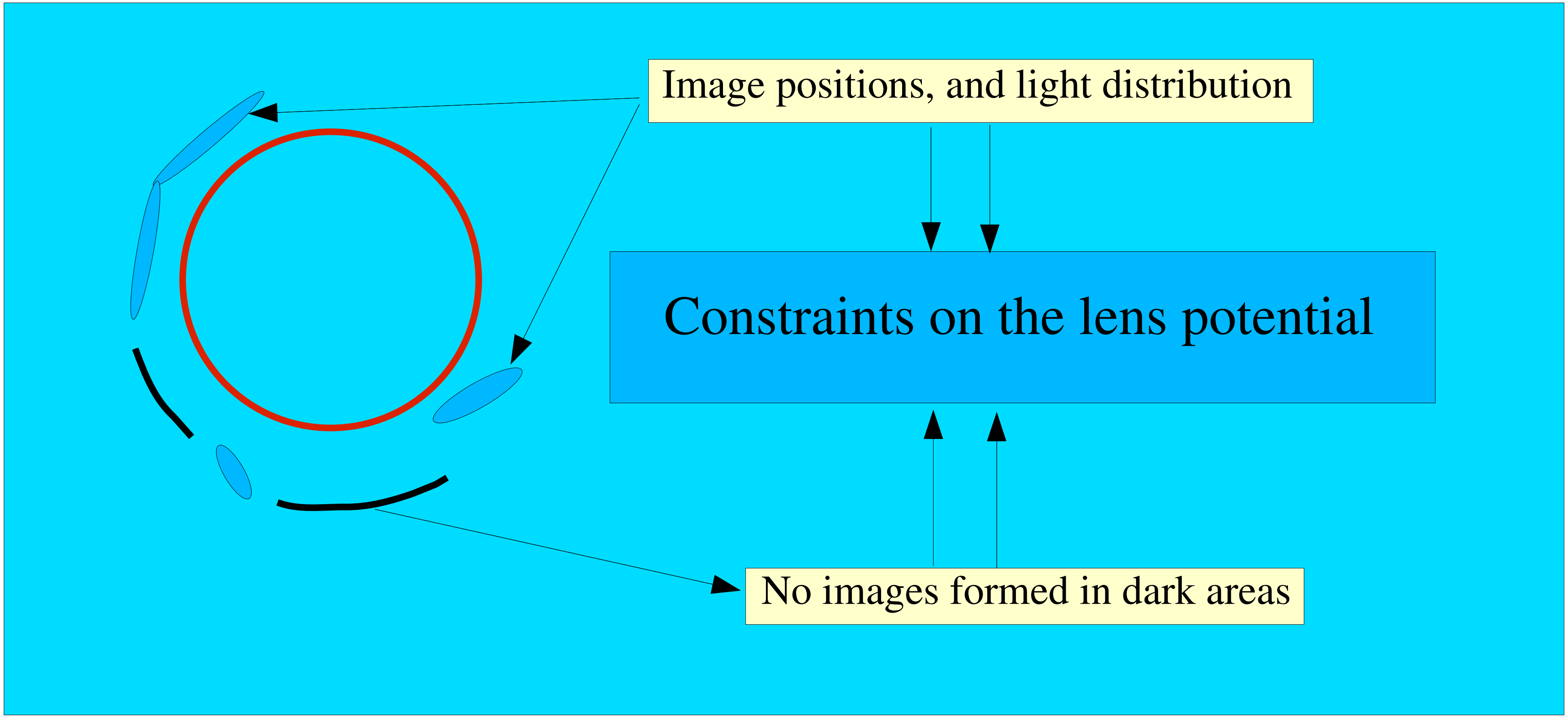
SL2SJ021408-053532





$$dr = \frac{1}{\kappa_2}\left(\tilde{f}_1 \pm \sqrt{R_0^2 - \left(\frac{d\,\tilde{f}_0}{d\theta}\right)^2}\right)$$

Gravitational lenses: degeneracy

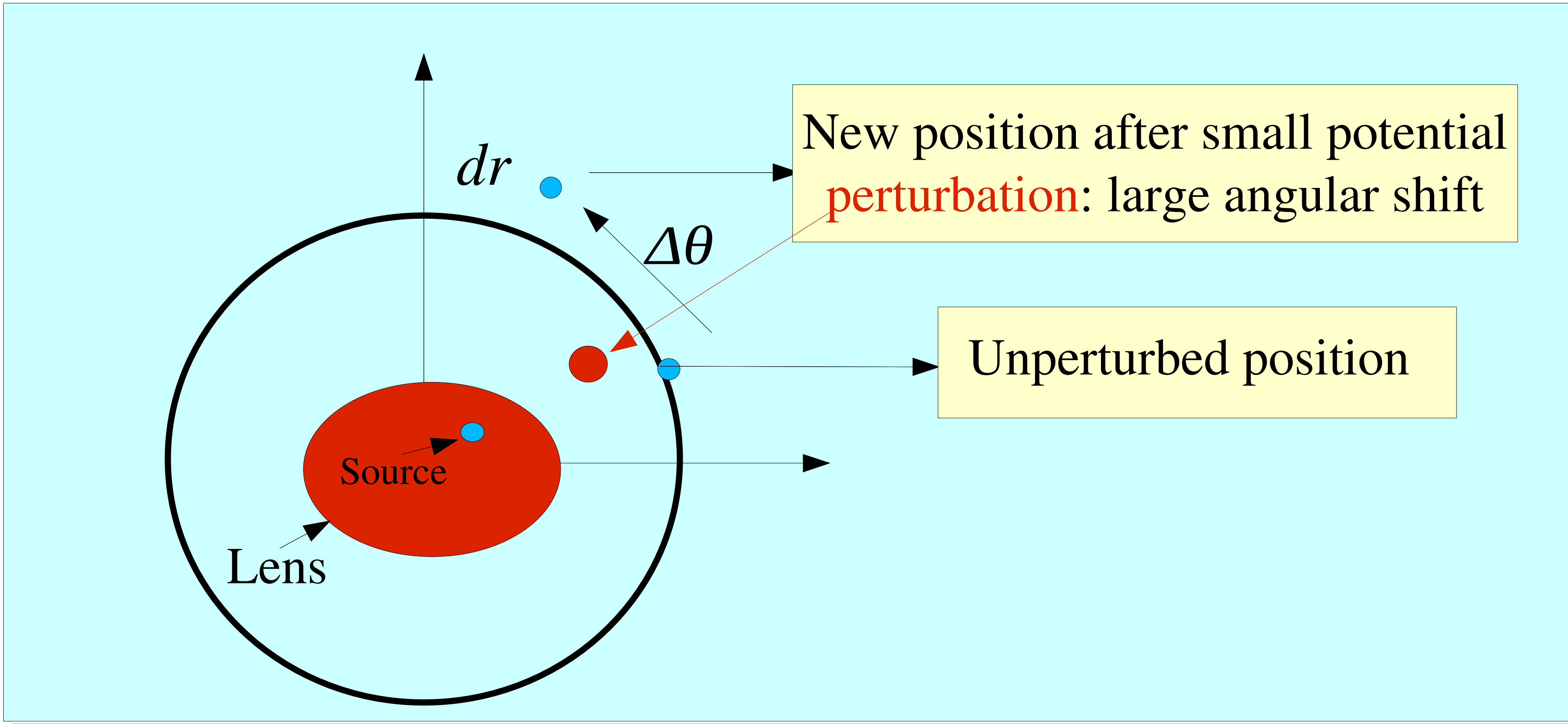


Potential is constrained in the neighborhood of the critical circle only

A familly of models meet such constraint: model degeneracy

Non degenerate representation: perturbative theory ?

Perturbative ? but strong lensing is highly non-linear ?



Response to a small pertrubation very large...hopeless ?
Actually $\Delta\theta$ is large but dr is small
(arcs forms near the critical circle)

Nearly round solution

$$\phi(r,\theta)=\phi_0(r)+\epsilon\psi(r,\theta)$$

$$r_S \equiv \epsilon r_S$$

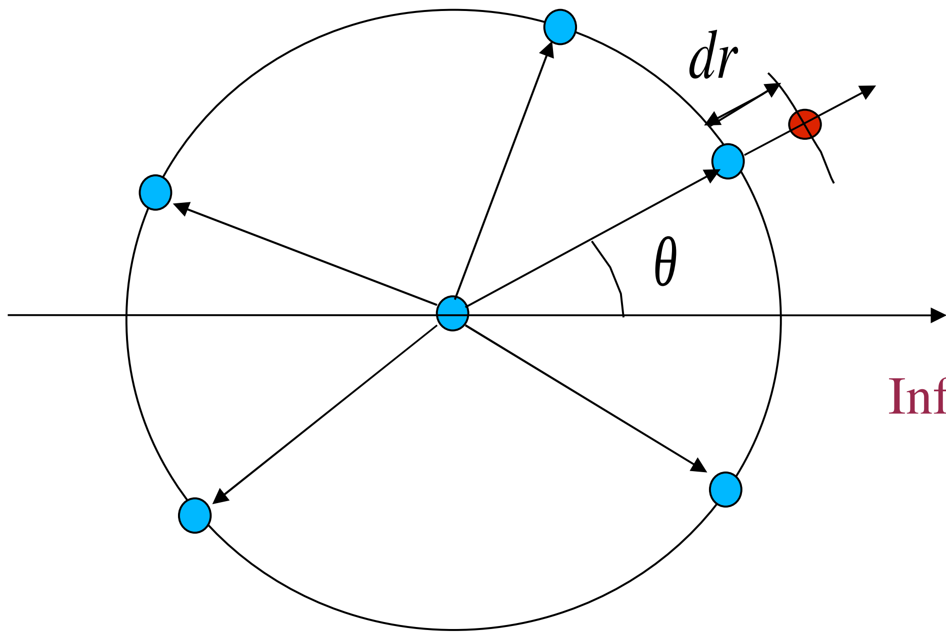
Problem how to derive the response to the perturbation ?

Attempt by Blandford & Kovner (1988): found a perturbative equation for caustics, but could not derive any equation for images reconstruction (try to make geometrical construction from caustics) -

Generic problem with prediction of image positions ($\Delta\theta$ large)

...something's missing...

The singular perturbative solution



Perfect ring singularity

Infinite number of images of a point

The perfect ring situation: a point centered at the center of a cicular lens
An image of the point exists at all θ -> considering any perturbed point, there is always an un-perturbed point at the same θ - Problem of large $\boldsymbol{\Delta\theta}$ is solved. Only dr has to be estimated.

The perturbative equations

(Alard 2007)

Working near perfect ring (r=1) $r = 1 + \epsilon\, dr$

Perturbation of perfect ring

$$\varphi = \varphi_0(r) + \epsilon\,\psi(r,\theta) = a_0 + 1 + \epsilon\, dr + \epsilon\left(f_0(\theta) + \epsilon\, f_1(\theta)\, dr\right)$$

$$\vec{r}_S \equiv \varepsilon\, \vec{r}_S$$

Perturbative response: dr given by perturbative lens equation

$$\vec{r}_S = \vec{r} - \overrightarrow{\nabla}\varphi$$

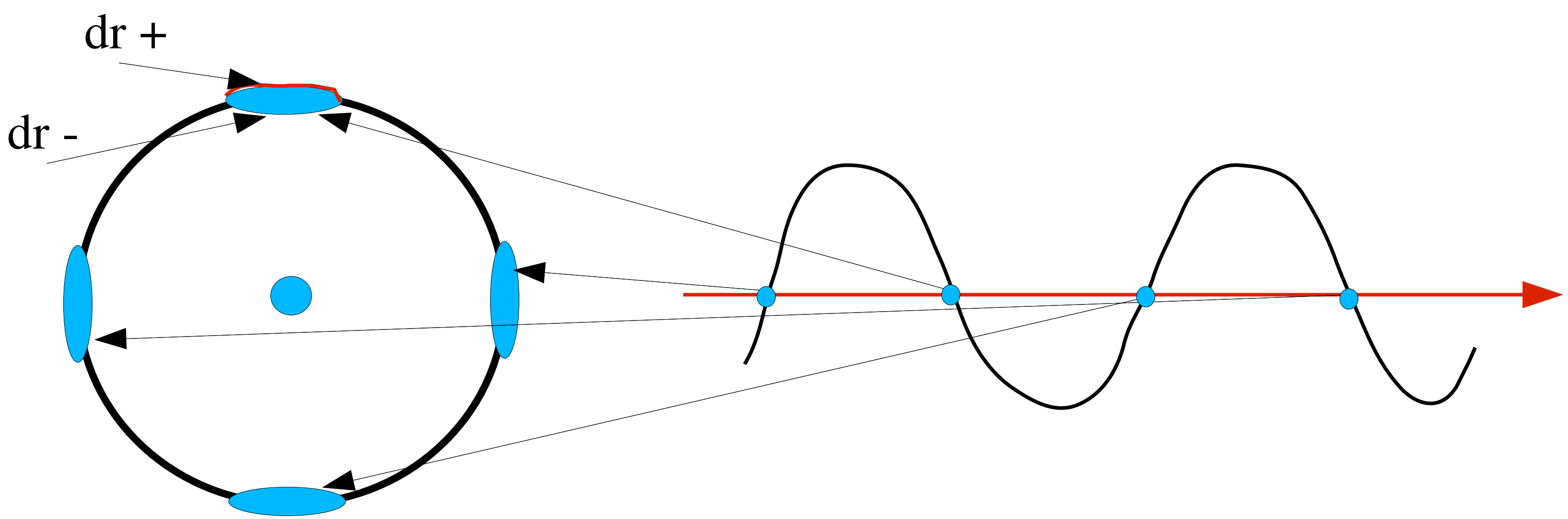
$$\vec{r}_s = \left(\kappa_2\, dr - f_1\right)\vec{u}_r - \frac{d f_0}{d\theta}\,\vec{u}_\theta$$

The perturbative response is entirely controled by 2 functions of θ, $f_0(\theta)$ and $f_1(\theta)$ the 2 first derivatives of the potential on the unit circle

The circular solution

$$\vec{r}_S = \vec{\tilde{r}}_S + \vec{r}_0 \quad , \quad \tilde{f}_i = f_i + x_0 \cos\theta + y_0 \sin\theta \;\rightarrow\; \vec{\tilde{r}}_S = (\kappa_2 dr - \tilde{f}_1)\vec{u}_r - \frac{d\tilde{f}_0}{d\theta}\vec{u}_\theta$$

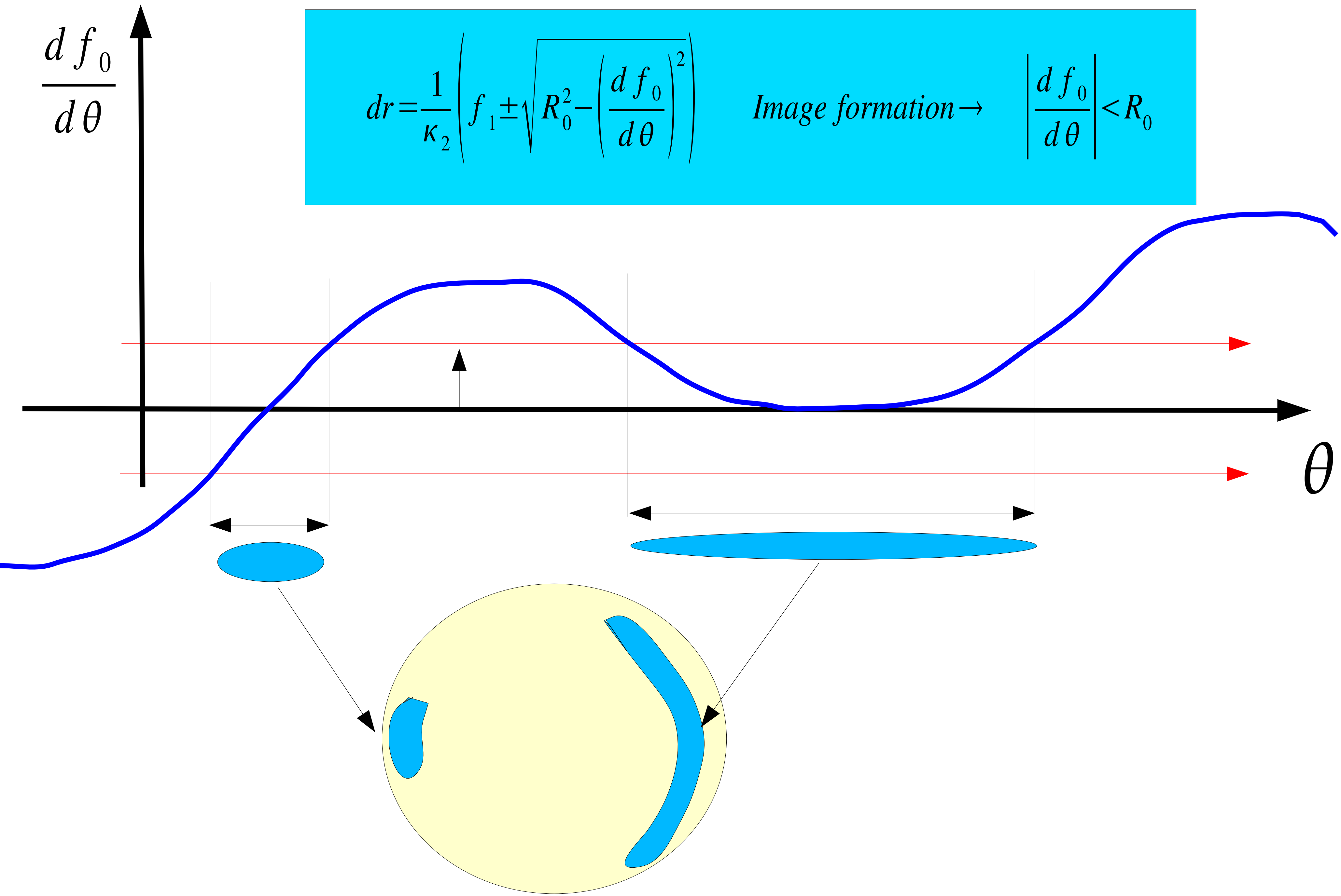
$$\left\|\vec{r}_S\right\| = R_0 \quad \longrightarrow \quad dr = \frac{1}{\kappa_2}\left(\tilde{f}_1 \pm \sqrt{R_0^2 - \left(\frac{d\tilde{f}_0}{d\theta}\right)^2}\right) \qquad \textit{Image formation} \rightarrow \quad \left|\frac{d\tilde{f}_0}{d\theta}\right| < R_0$$



$$\phi(r,\theta) \equiv F(r(1-\eta/2\cos 2\theta)) \simeq F(r) - \eta/2 \frac{dF}{dr}(r)\cos 2\theta \;\rightarrow\rightarrow\; \frac{df_0}{d\theta} = \eta \sin 2\theta - x_0 \sin\theta + y_0 \cos\theta$$

Image formation in perturbative theory

$$dr=\frac{1}{\kappa_2}\left(f_1\pm\sqrt{R_0^2-\left(\frac{d f_0}{d\theta}\right)^2}\right) \qquad Image\ formation \rightarrow \qquad \left|\frac{d f_0}{d\theta}\right|<R_0$$



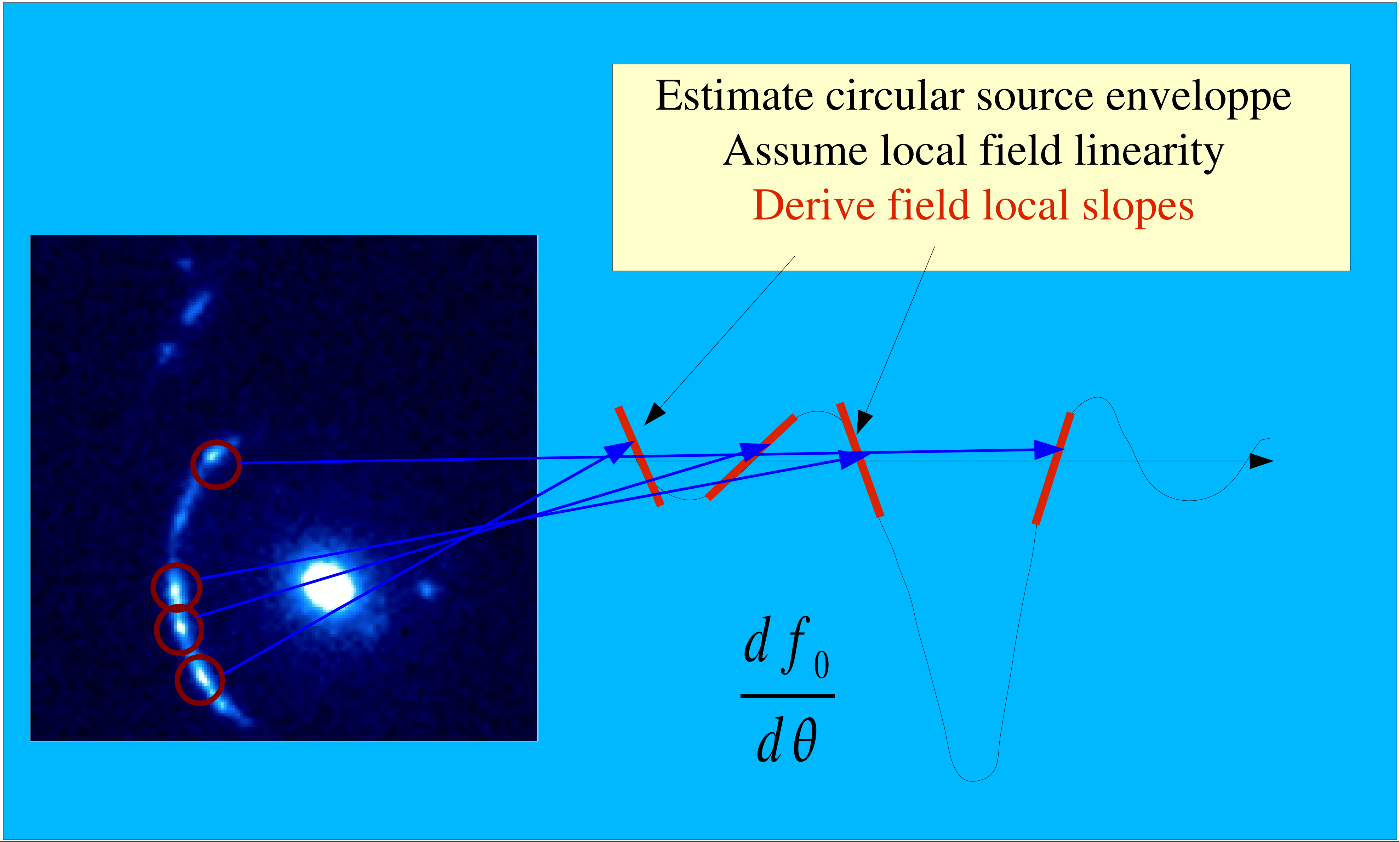
Accuracy of 1rst order pertubative theory
CDM halo's as lens -> accuracy better than 1 %
(Peirani etal. 2008)

Lens inversion is reduced to the reconstruction of the perturbative fields

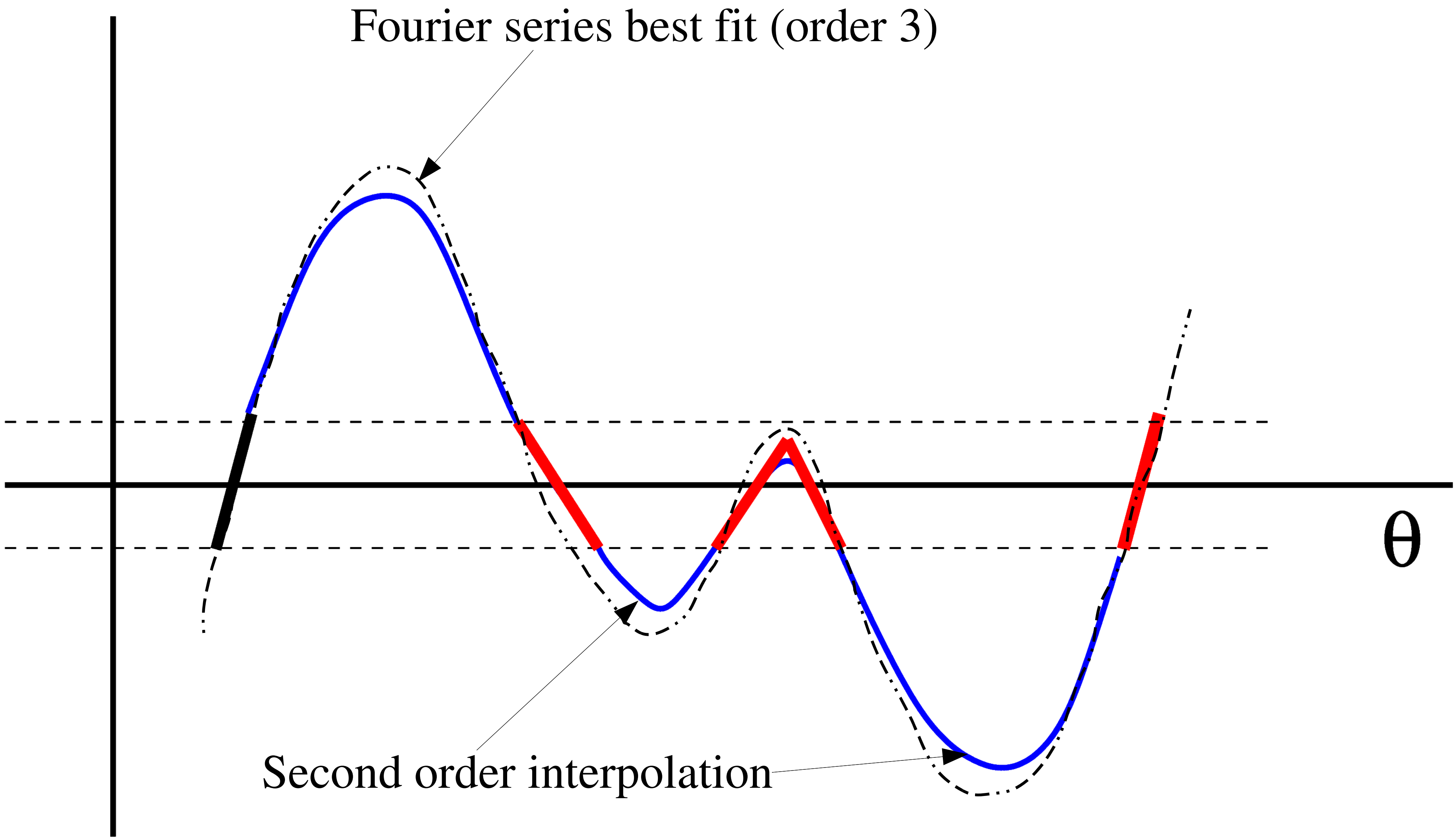
A familly of models corresponds to a given set of perturbative fields
Degeneracy is illustrated but in principle solved provided the perturbative field reconstruction is unique

An application to SL2S02176-0513l

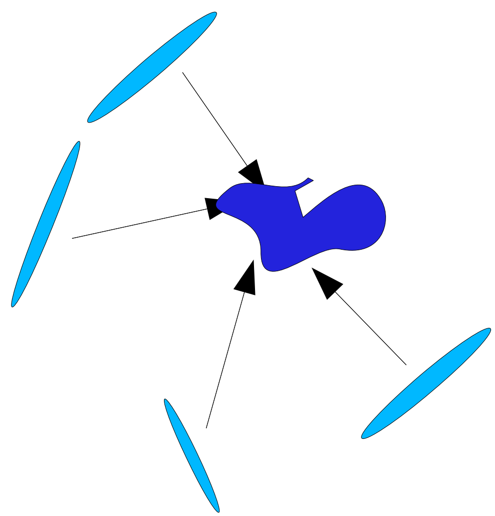
Perturbative fields reconstruction



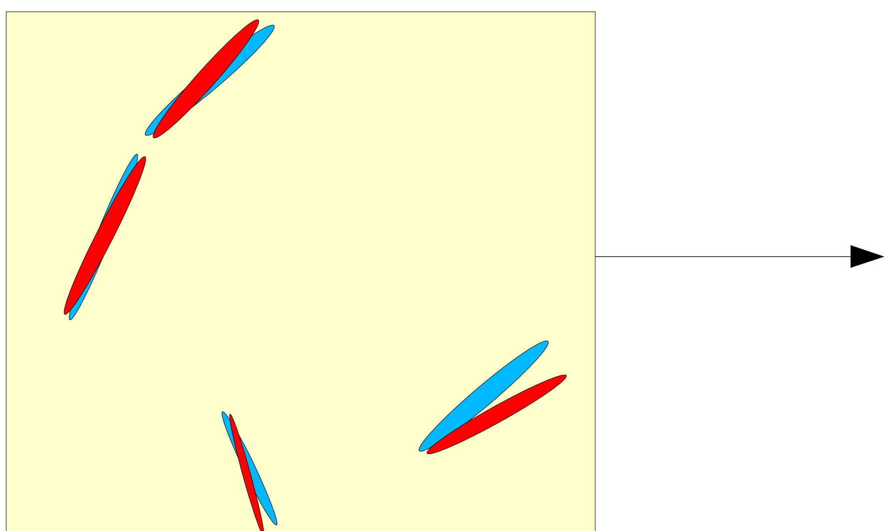
For circular source f_1 is the mean image position


Fourier series best fit (order 3)
θ
Second order interpolation

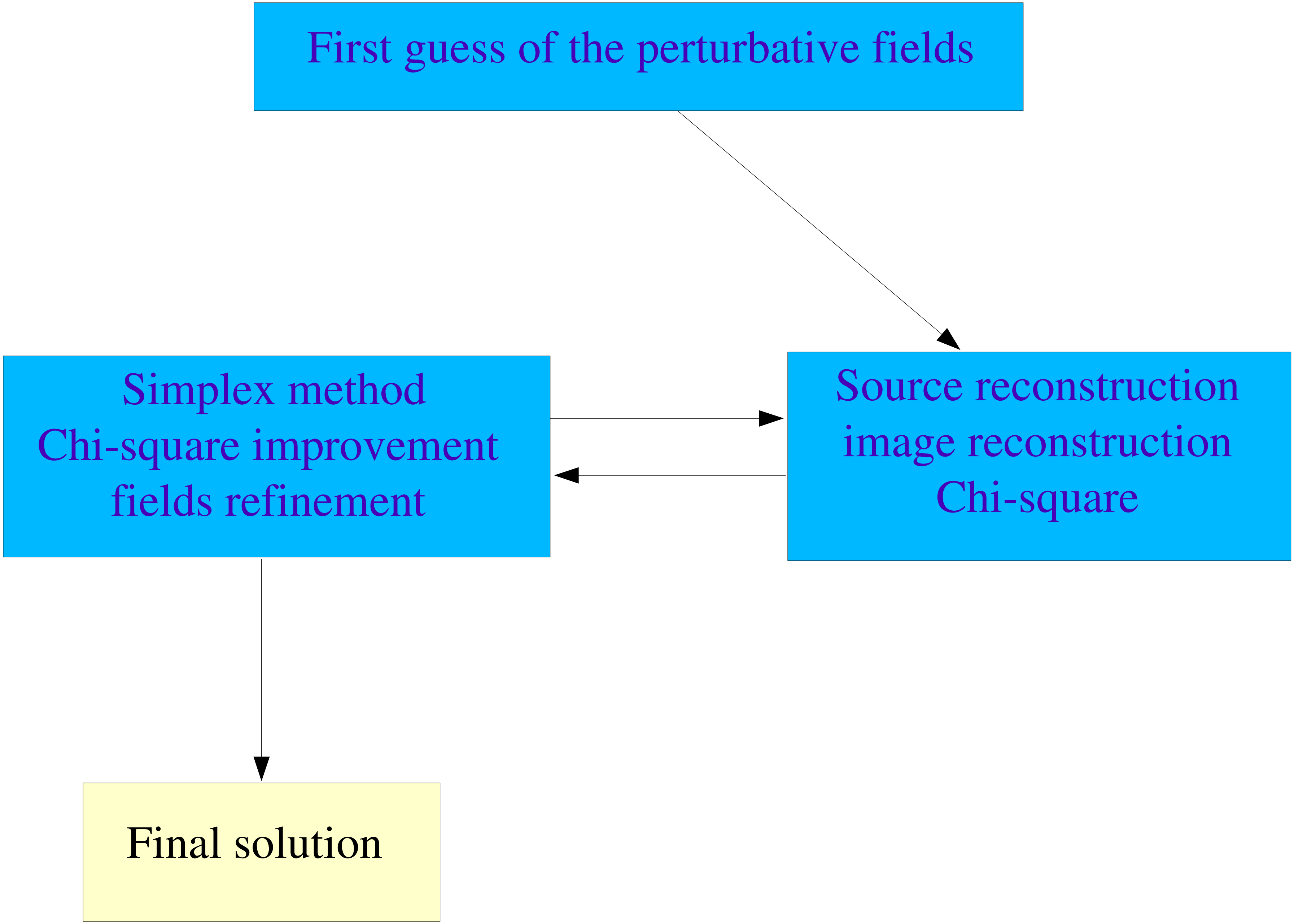
Fitting the fields to reproduce HST data

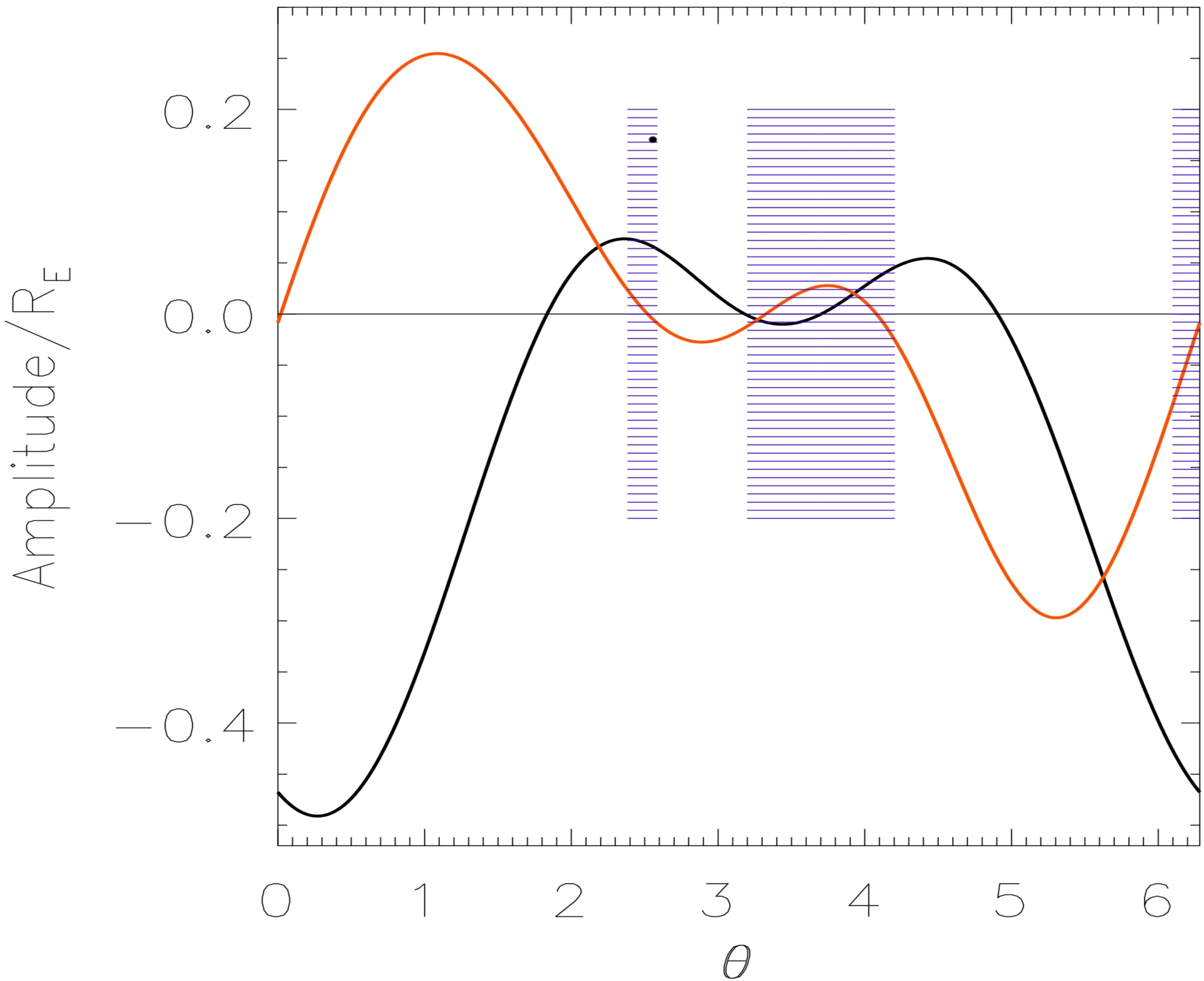


Ray-trace the source to obtain images, and convolve with HST PSF

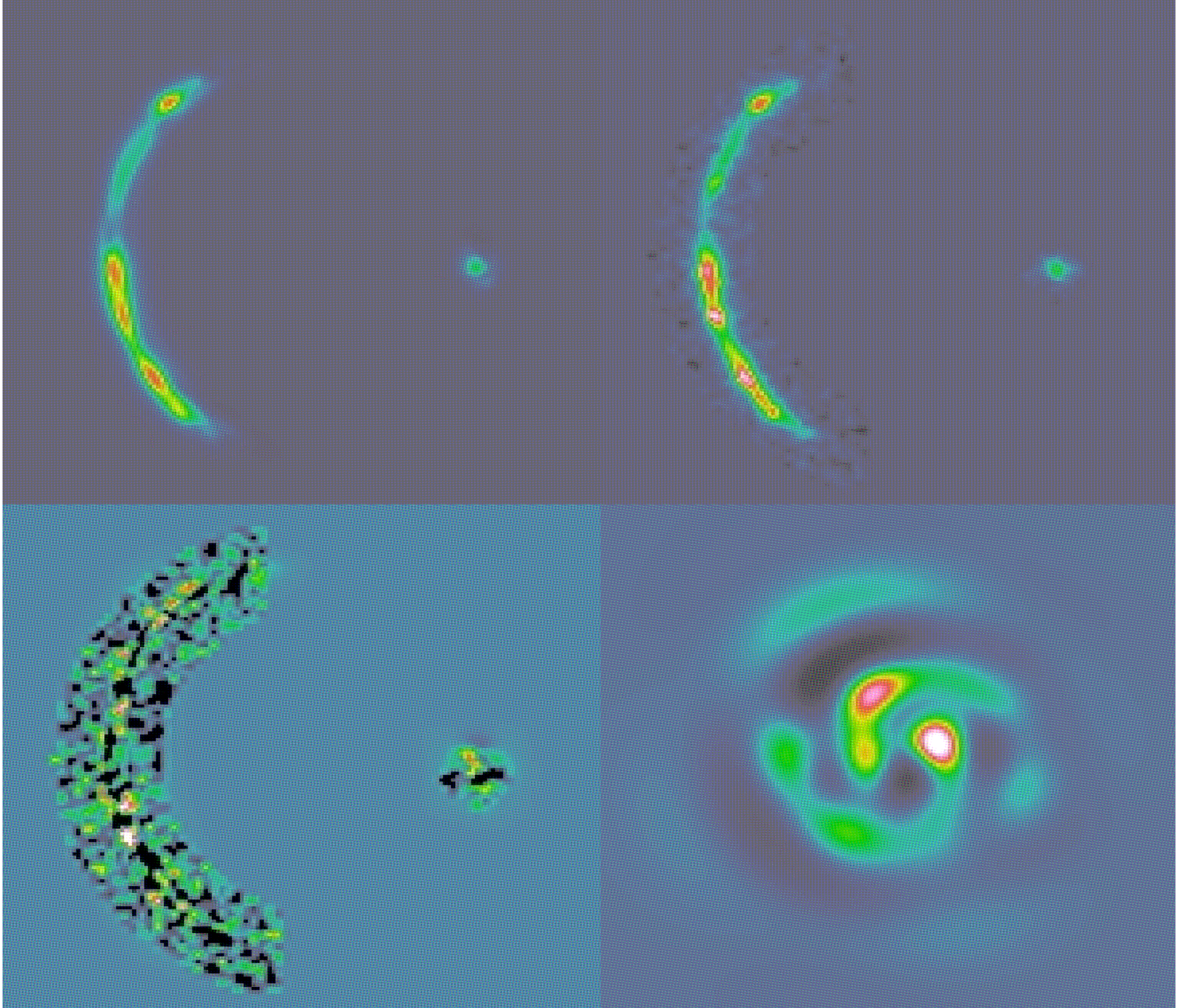


Chi-Square estimation

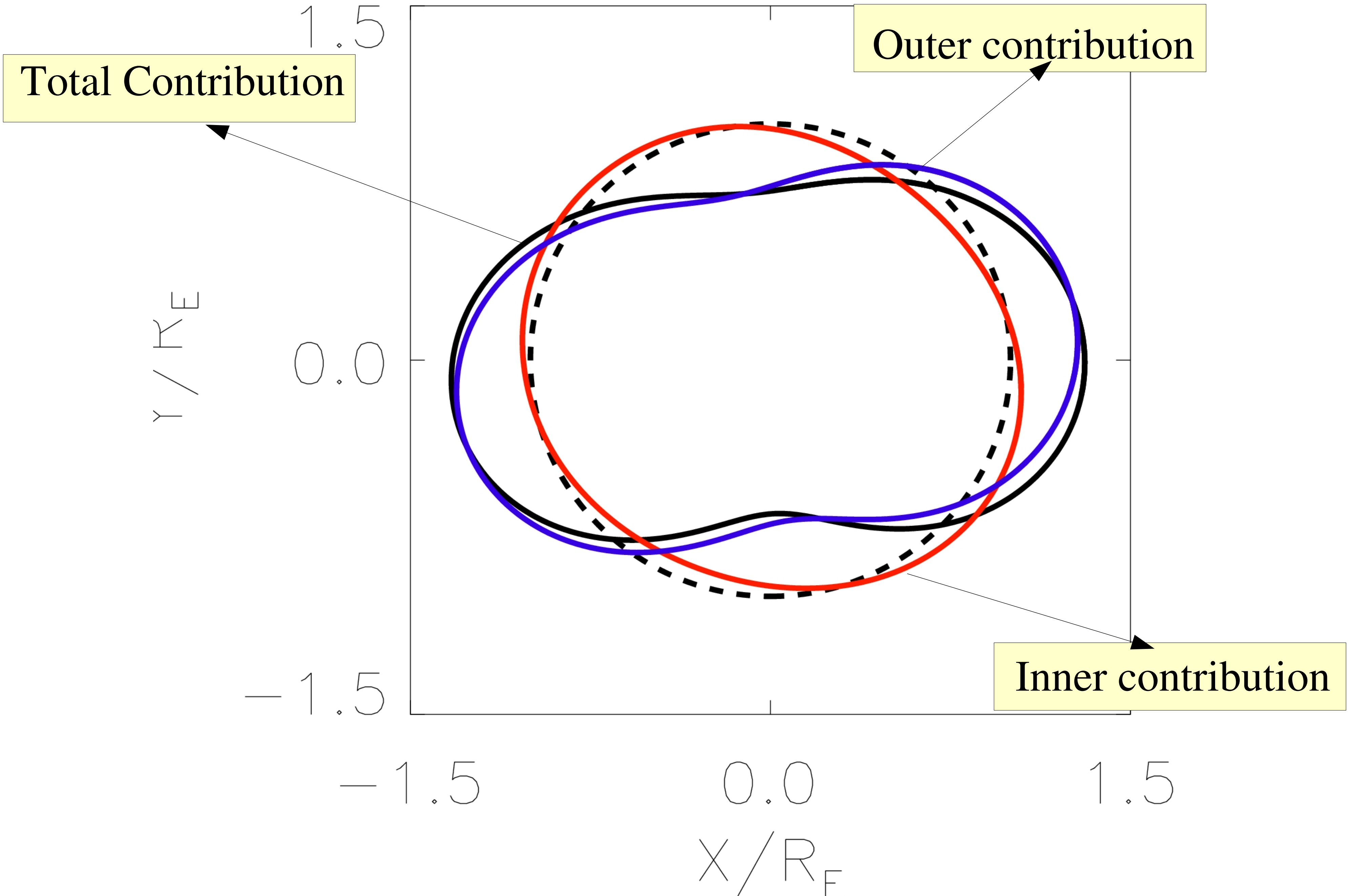

First guess of the perturbative fields
Simplex method
Chi-square improvement
fields refinement
Source reconstruction
image reconstruction
Chi-square
Final solution



0.2
0.0
−0.2
−0.4
Amplitude/R_E
0
1
2
3
4
5
6
θ



Potential iso-contours



Inner versus outer contributions

$$\phi = \sum_n \frac{a_n(r)}{r^n}\cos n\theta + \frac{b_n(r)}{r^n}\sin n\theta + c_n(r)\, r^n \cos n\theta + d_n(r)\, r^n \sin n\theta$$

$$a_n(r) = \frac{1}{2\pi n}\int_0^{2\pi}\int_0^r \sigma(u,v)\ \cos nv\ \ u^{n+1}\, du\, dv$$

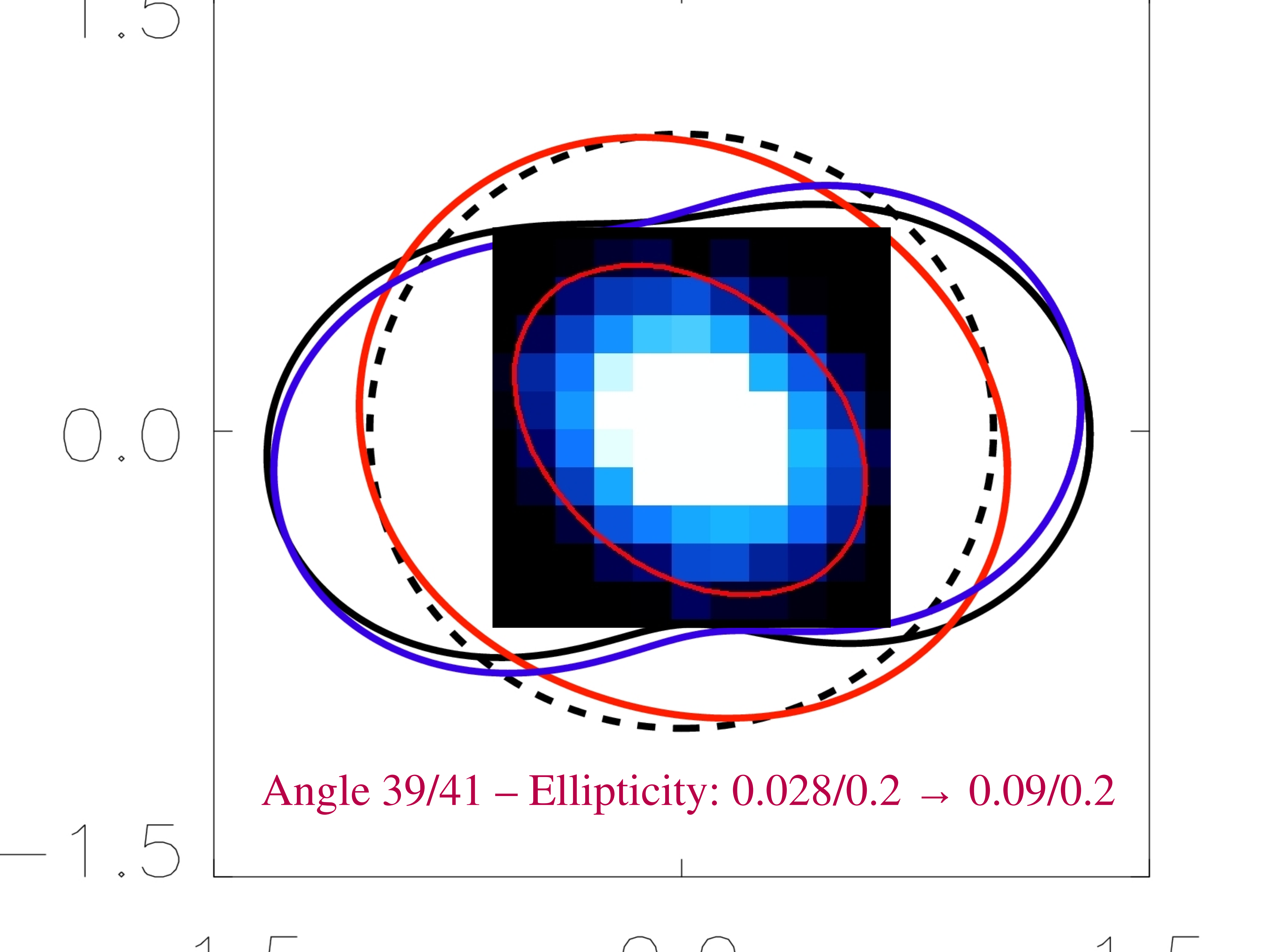
$$b_n(r) = \frac{1}{2\pi n}\int_0^{2\pi}\int_0^r \sigma(u,v)\ \sin nv\ \ u^{n+1}\, du\, dv$$

$$c_n(r) = \frac{1}{2\pi n}\int_0^{2\pi}\int_1^\infty \sigma(u,v)\ \cos nv\ \ u^{1-n}\, du\, dv$$

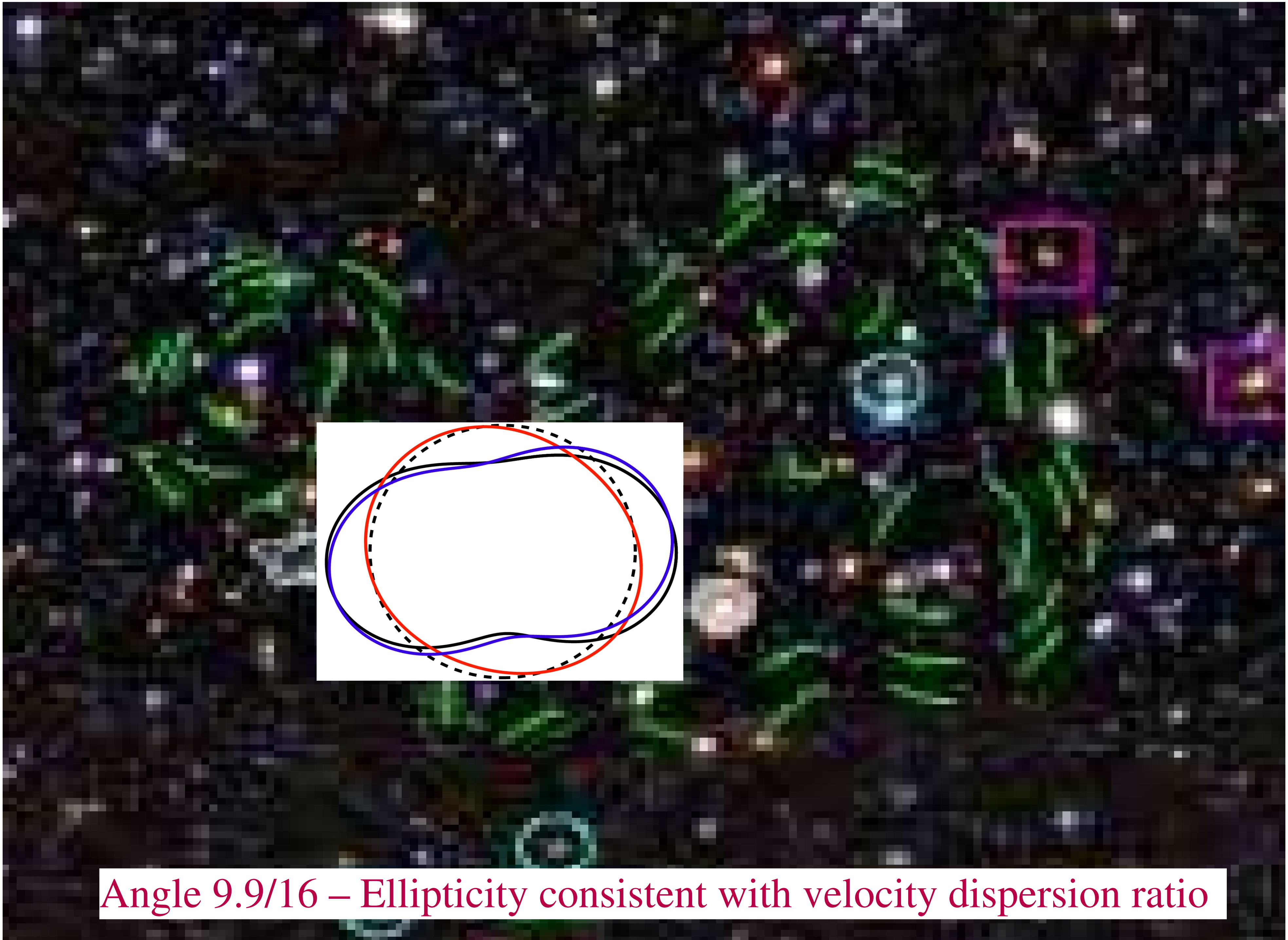
$$d_n(r) = \frac{1}{2\pi n}\int_0^{2\pi}\int_1^\infty \sigma(u,v)\ \sin nv\ \ u^{1-n}\, du\, dv$$

$$f_1 = \sum_n n(a_n - c_n)\cos n\theta + n(b_n - d_n)\sin n\theta$$

$$\frac{d f_0}{d\theta} = \sum_n -n(b_n + d_n)\cos n\theta + n(a_n + c_n)\sin n\theta$$


1.5
0.0
−1.5
Angle 39/41 – Ellipticity: 0.028/0.2 → 0.09/0.2

Geach etal. 2007 small group (Subaru/XMM-Newton)

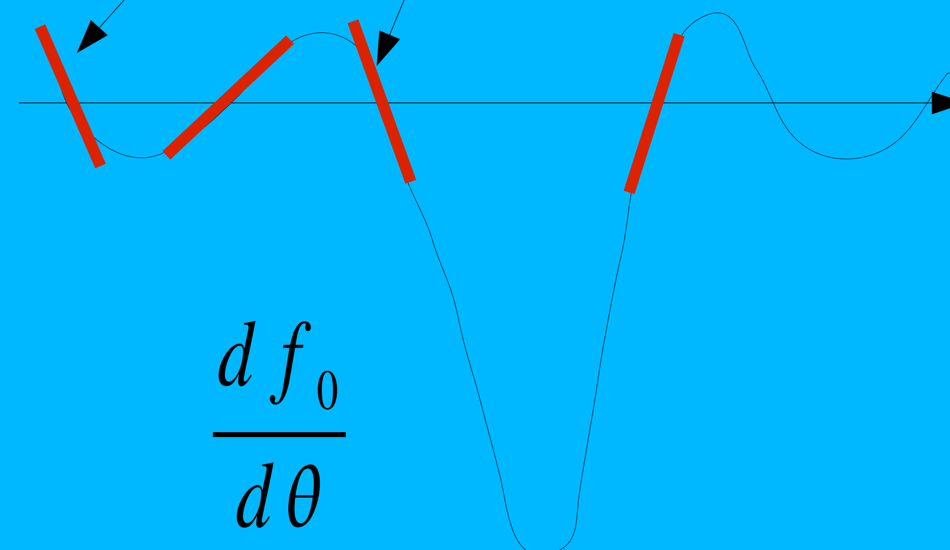


Perturbative fields reconstruction

Estimate circular source enveloppe

Assume local field linearity

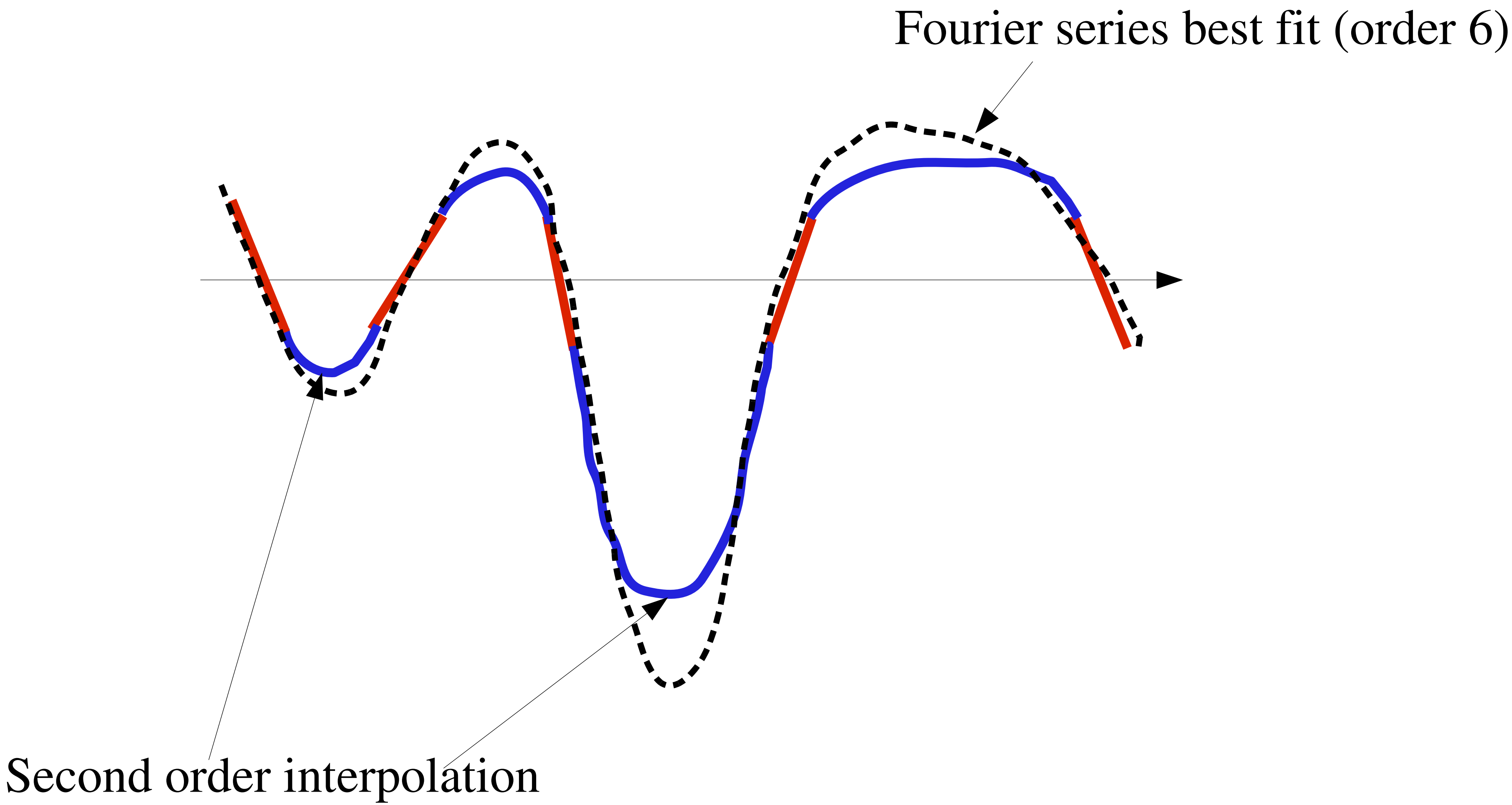
Derive field local slopes



$$\frac{d f_0}{d\theta}$$

For circular source f_1 is the mean image position

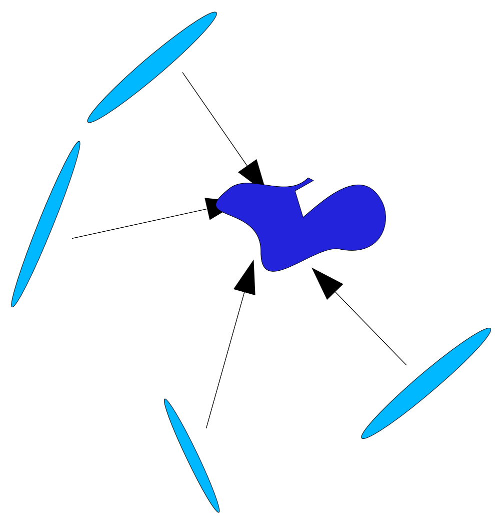
Complete first guess estimation of fields



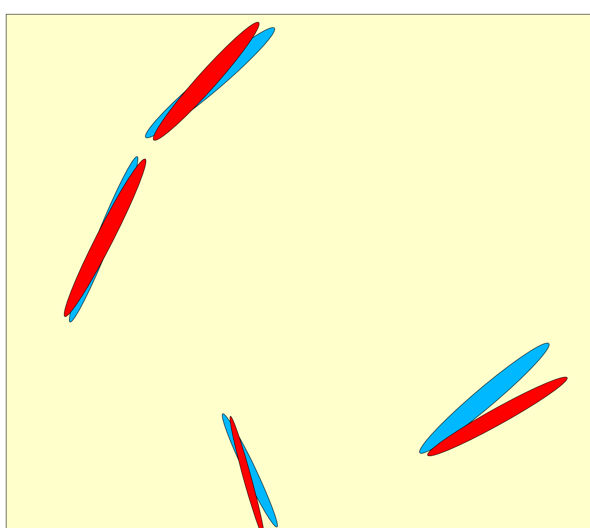
Fitting the fields to reproduce HST data

Source reconstruction

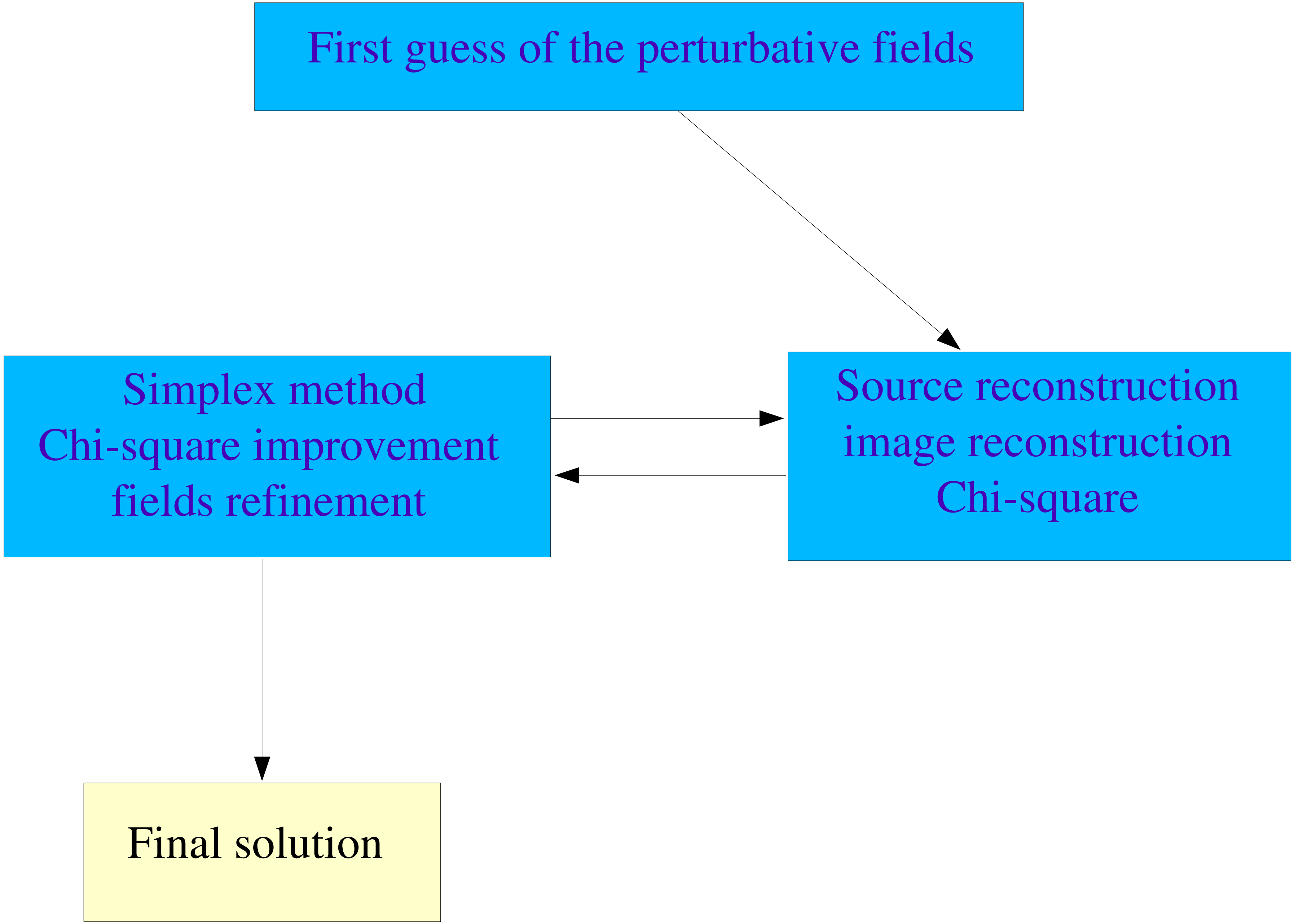
fold the image to the source plane using the perturbative lens equation

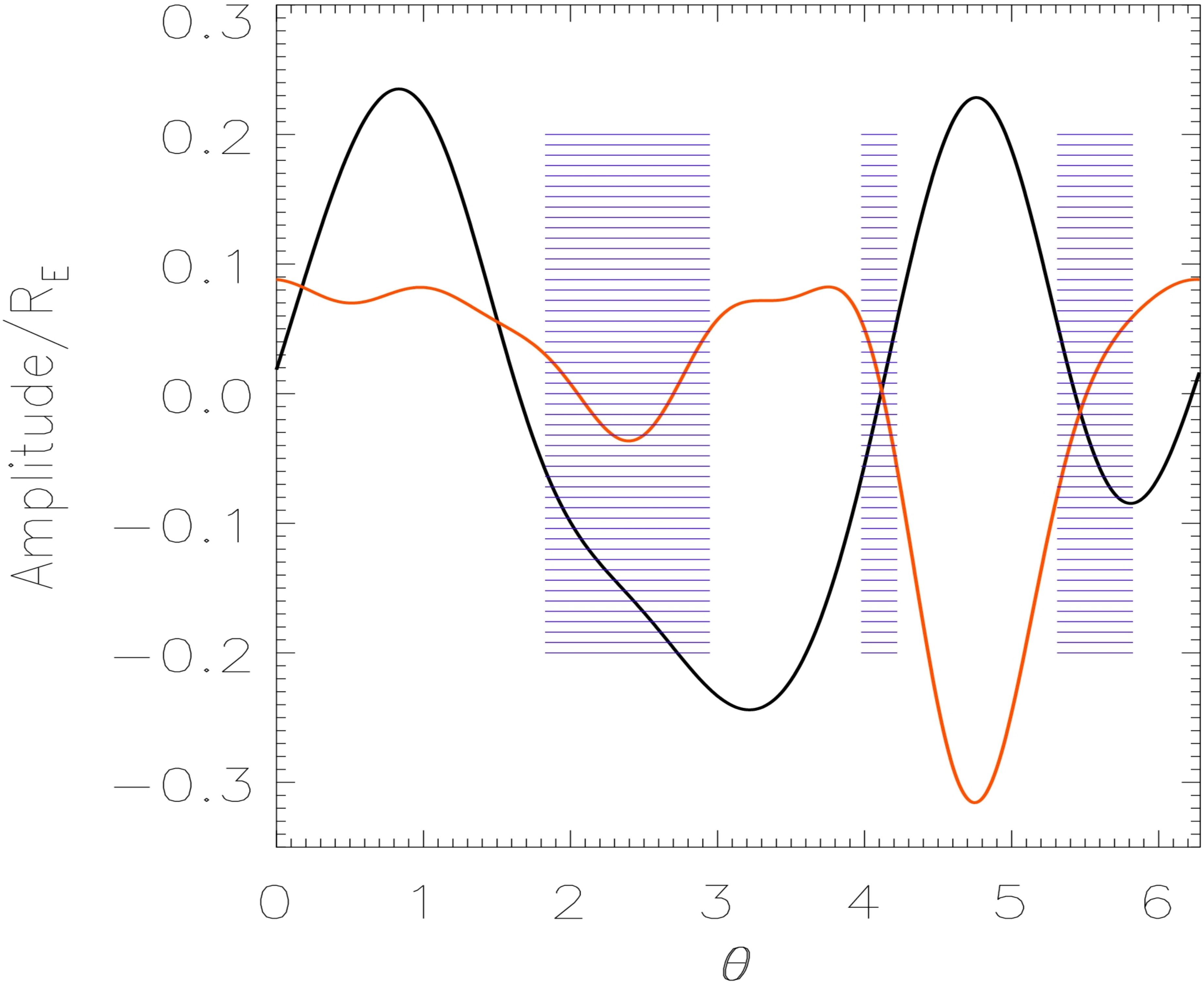


Ray-trace the source to obtain images, and convolve with HST PSF

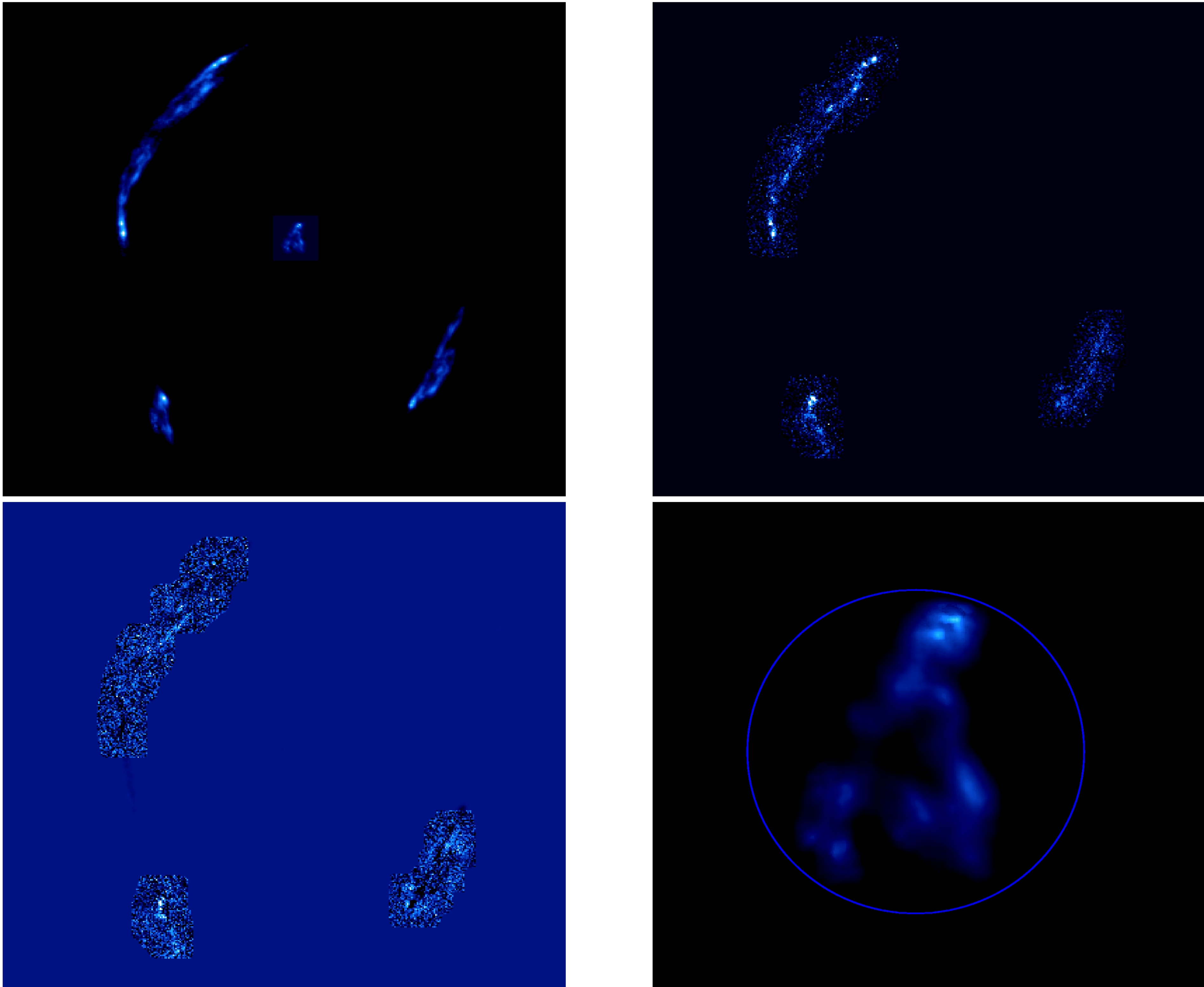


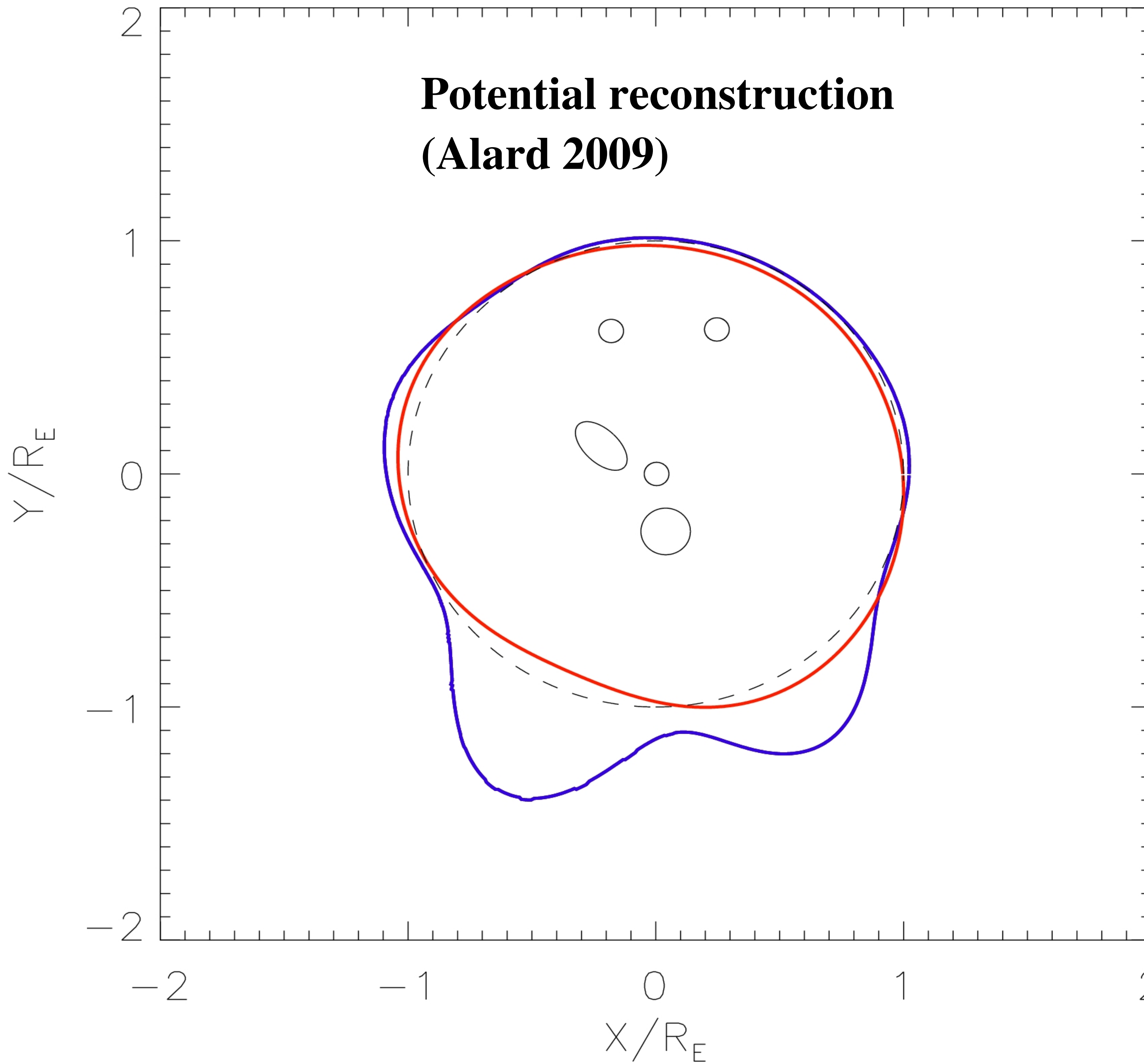
→ Chi-Square estimation

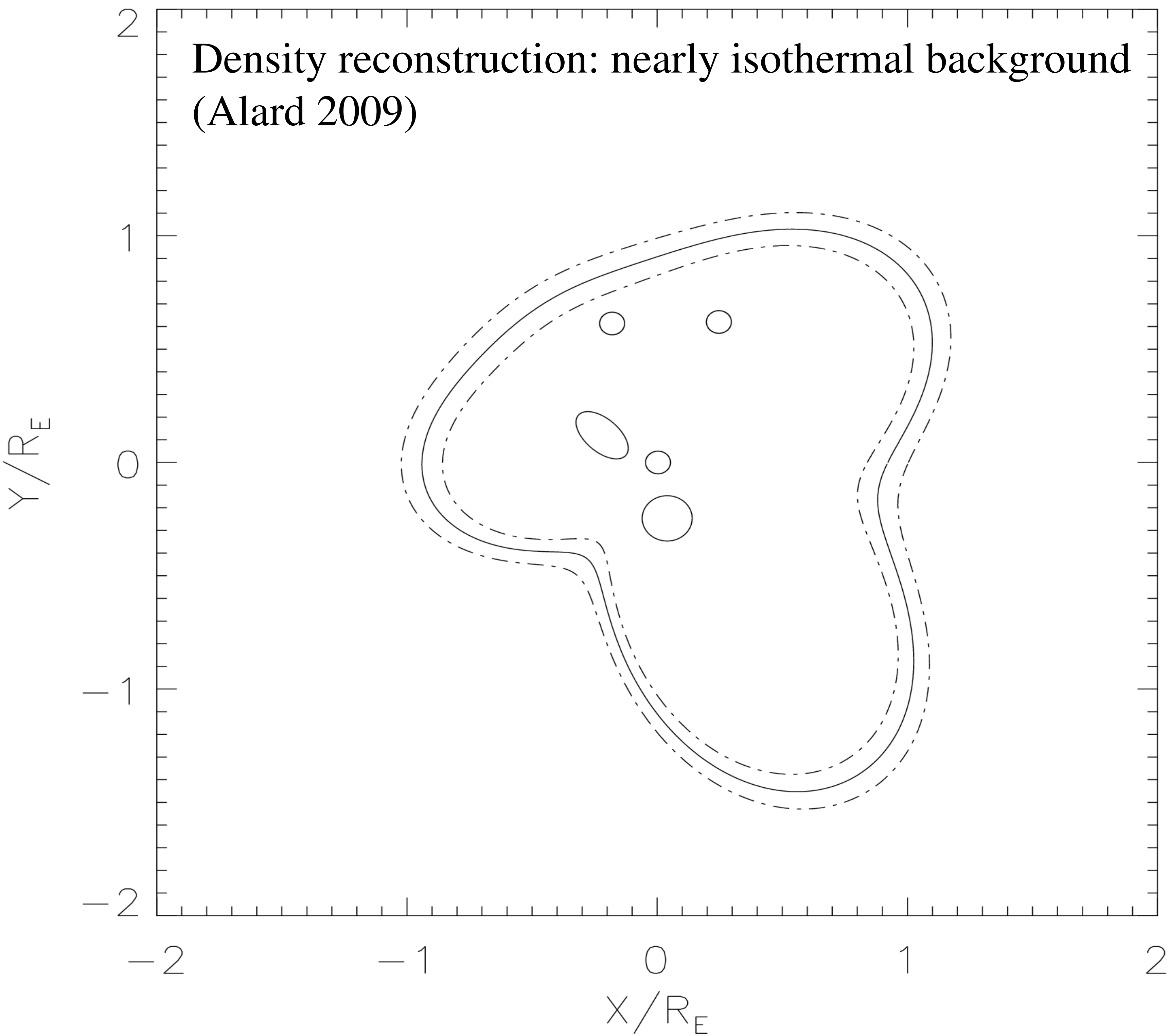

First guess of the perturbative fields
Simplex method
Chi-square improvement
fields refinement
Source reconstruction
image reconstruction
Chi-square
Final solution



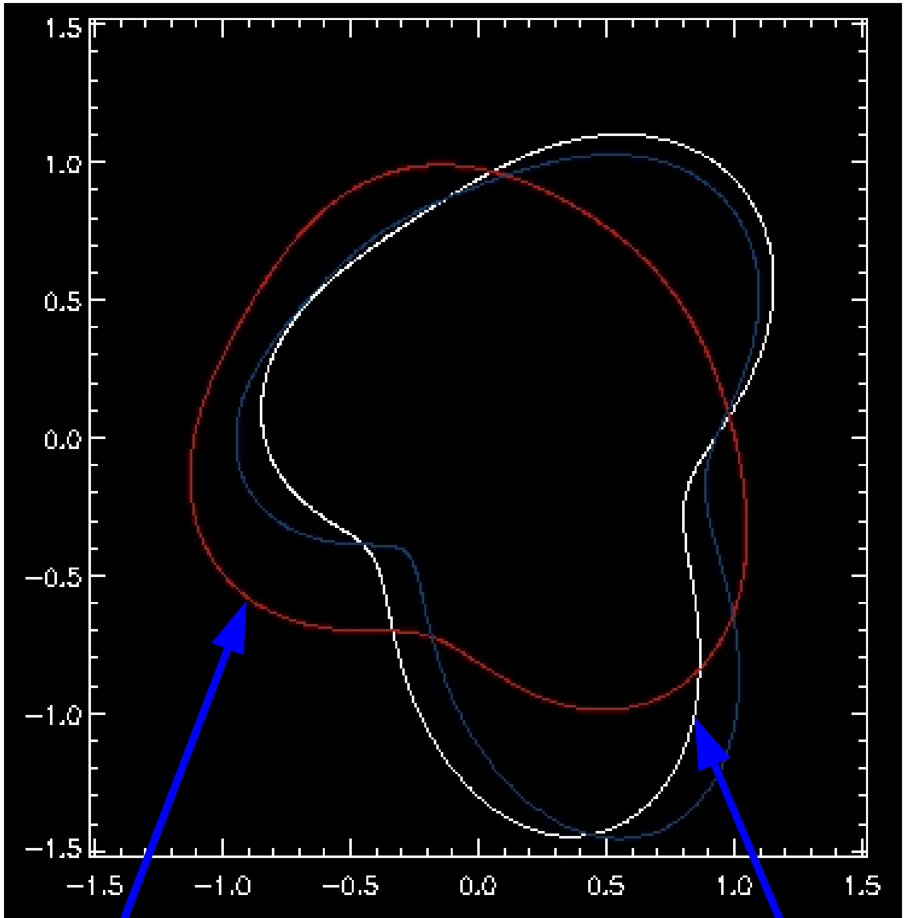
Amplitude/R_E
0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3
0
1
2
3
4
5
6
θ




Potential reconstruction
(Alard 2009)
2
1
0
−1
−2
Y/R_E
−2
−1
0
1
2
X/R_E


Density reconstruction: nearly isothermal background
(Alard 2009)
2
1
0
−1
−2
Y/R_E
−2
−1
0
1
2
X/R_E

Inner versus outer contributions



$$\phi = \sum_n \frac{a_n(r)}{r^n} \cos n\theta + \frac{b_n(r)}{r^n} \sin n\theta + c_n(r) r^n \cos n\theta + d_n(r) r^n \sin n\theta$$

$$a_n(r) = \frac{1}{2\pi n} \int_0^{2\pi} \int_0^r \sigma(u, v) \ \cos nv \ \ u^{n+1} du\, dv$$

$$b_n(r) = \frac{1}{2\pi n} \int_0^{2\pi} \int_0^r \sigma(u, v) \ \sin nv \ \ u^{n+1} du\, dv$$

$$c_n(r) = \frac{1}{2\pi n} \int_0^{2\pi} \int_1^\infty \sigma(u, v) \ \cos nv \ \ u^{1-n} du\, dv$$

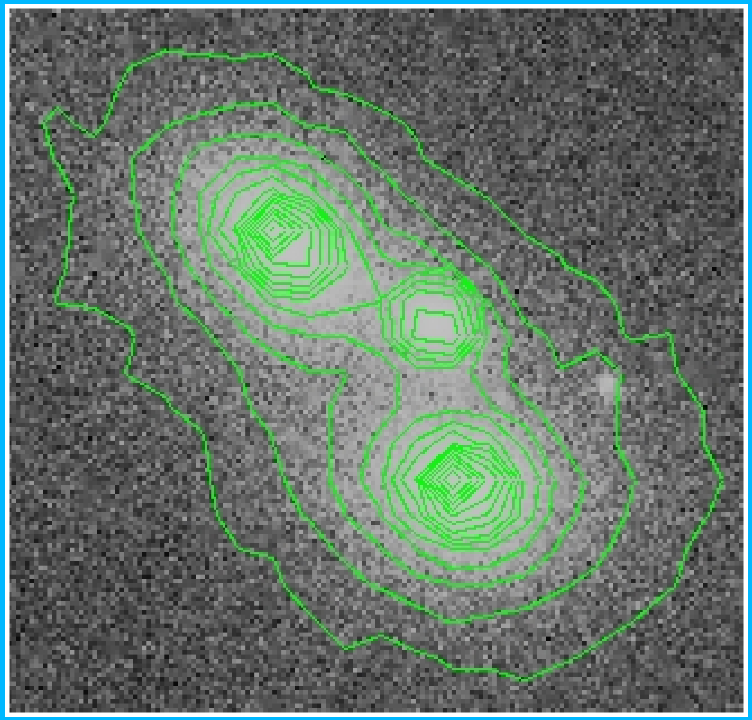
$$d_n(r) = \frac{1}{2\pi n} \int_0^{2\pi} \int_1^\infty \sigma(u, v) \ \sin nv \ \ u^{1-n} du\, dv$$

$$f_1 = \sum_n n(a_n - c_n) \cos n\theta + n(b_n - d_n) \sin n\theta$$

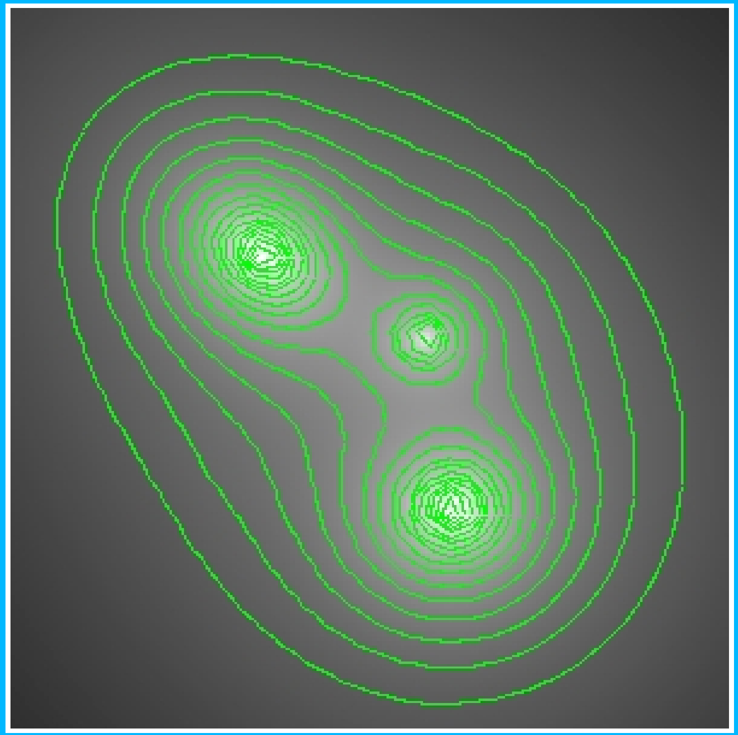
$$\frac{d f_0}{d\theta} = \sum_n -n(b_n + d_n) \cos n\theta + n(a_n + c_n) \sin n\theta$$

Inside density shape

Outside density shape



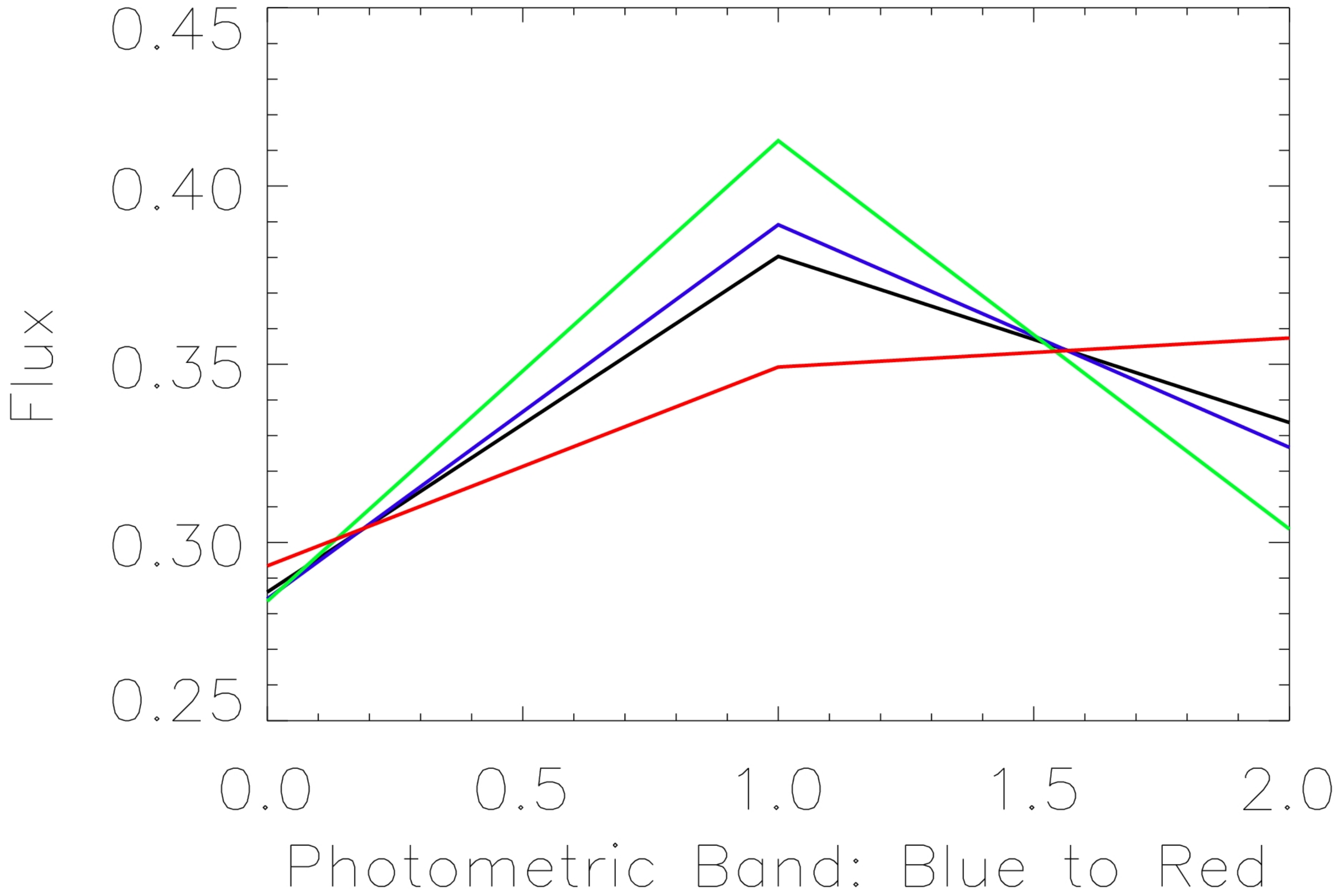
Iso-contours: original image

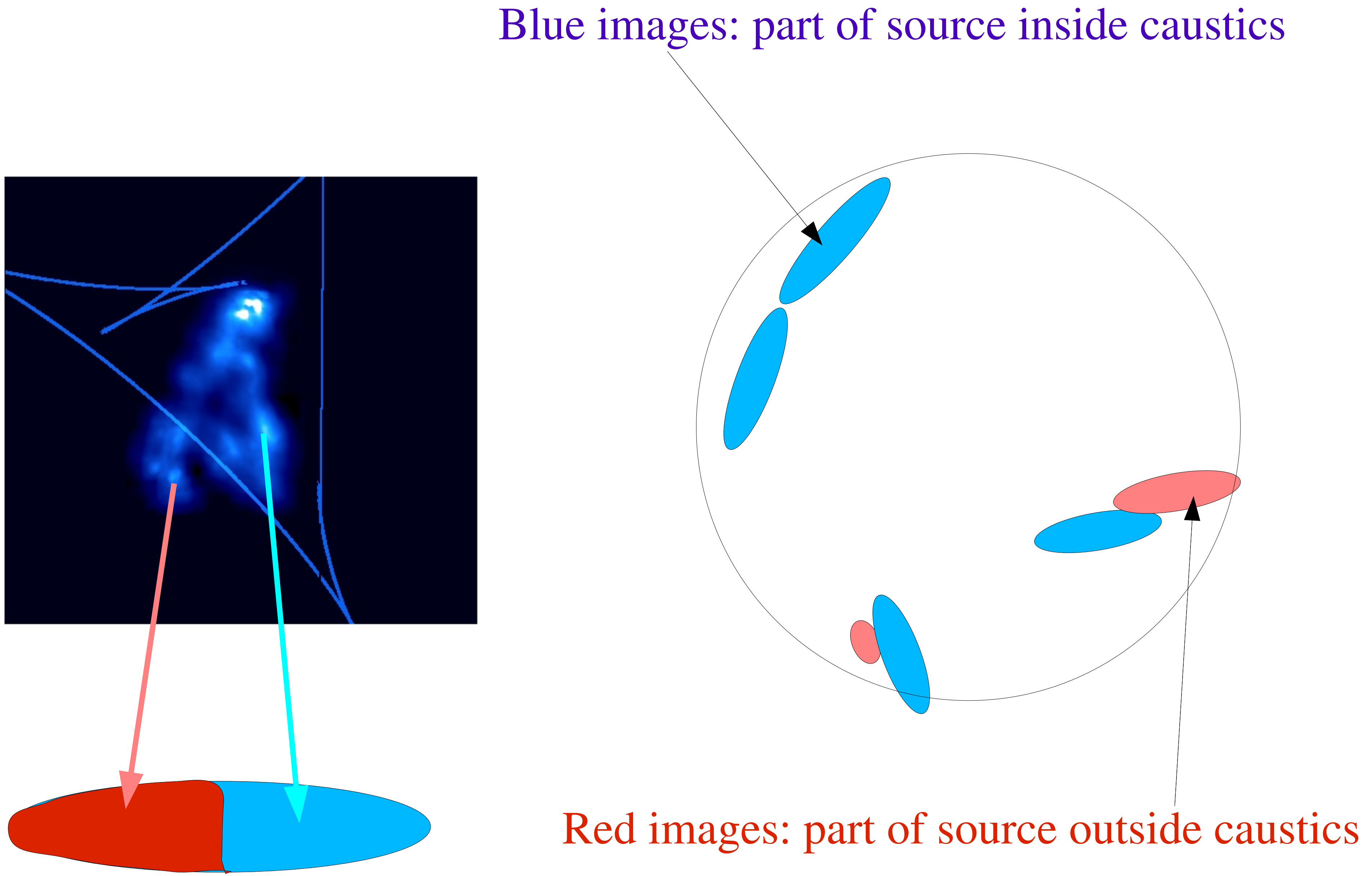


Iso-contours: model resconstruction profiles from isolated galaxies

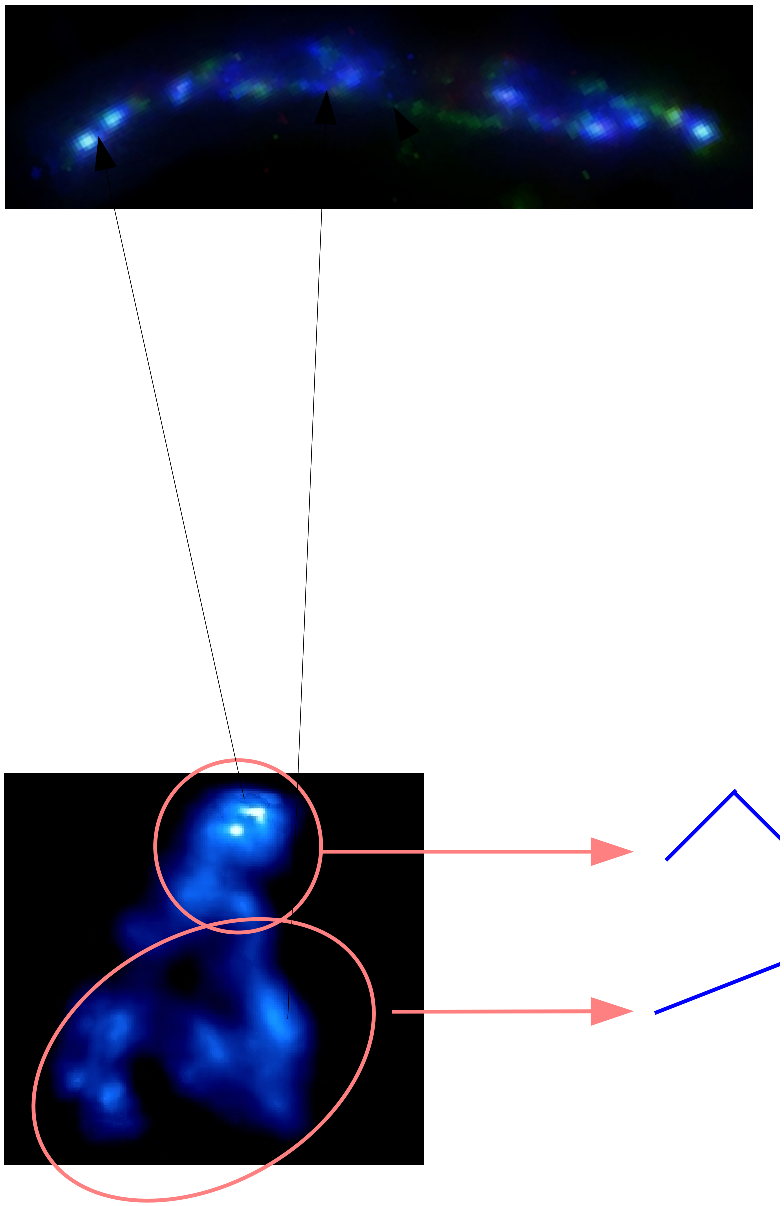
Conclusion: the 3 galaxies share a common outer halo
Obvious sign of merging process

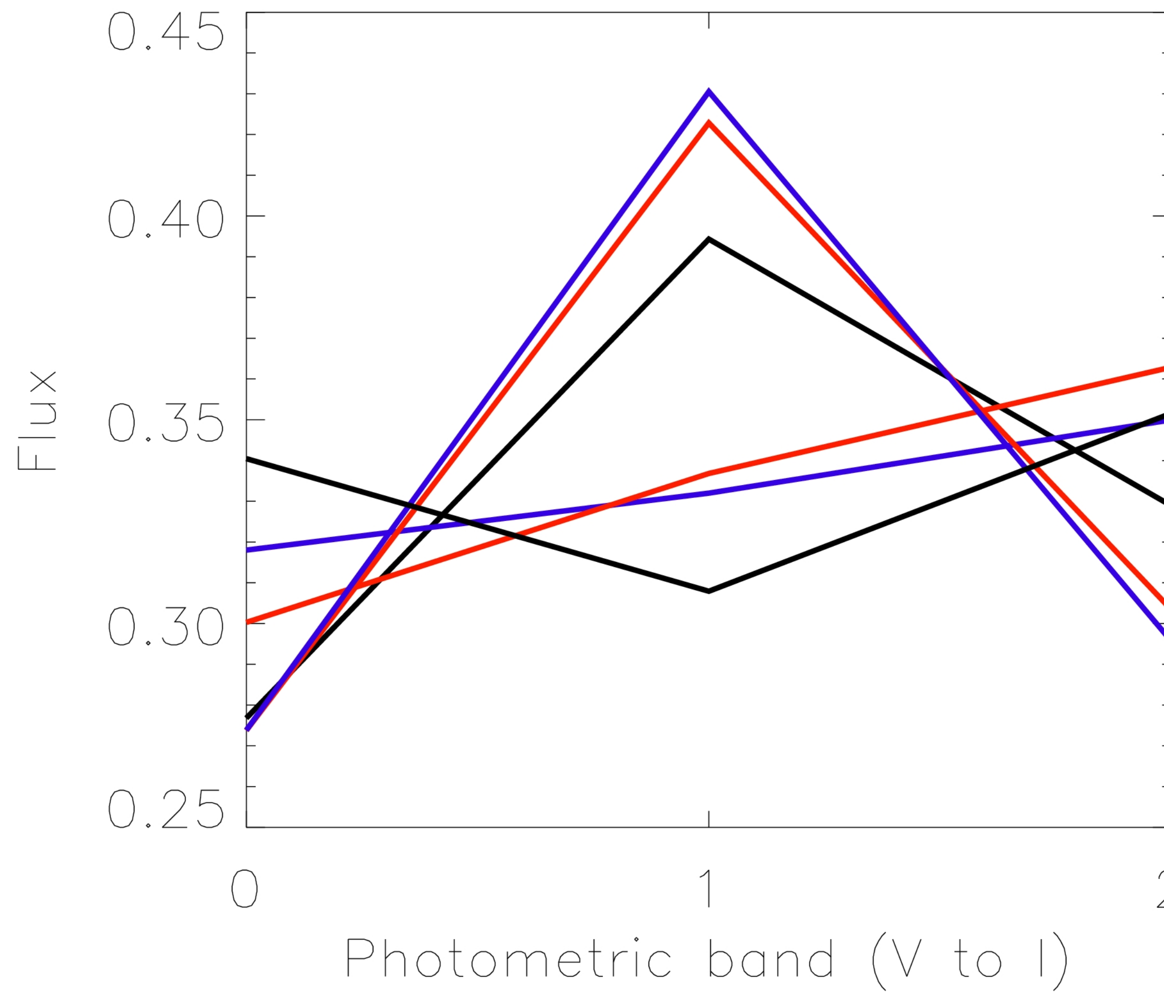
Color spectrum of the 4 images

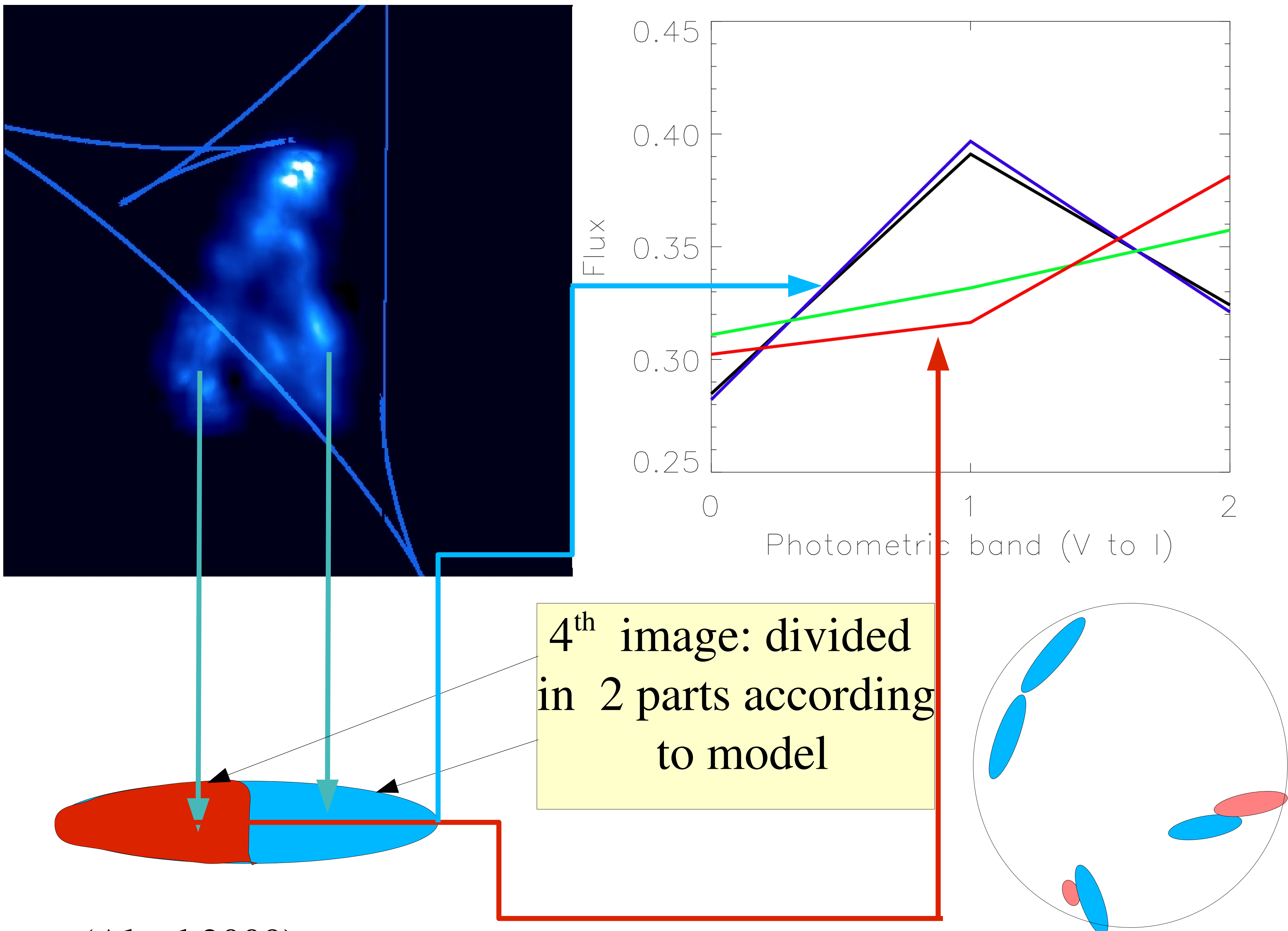



Blue images: part of source inside caustics
Red images: part of source outside caustics

Color spectrum of half images
(first 2 images)






0.45
0.40
0.35
0.30
0.25
Flux
0
1
2
Photometric band (V to I)
4th image: divided in 2 parts according to model

(Alard 2009)

Implications

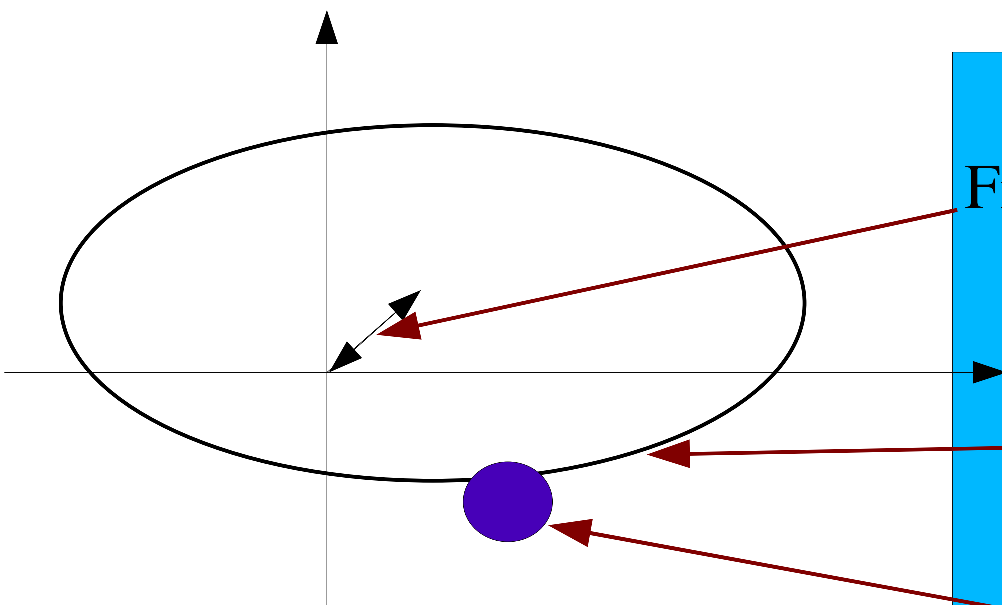
Previously: dark matter at the scale of clusters (weak lensing)
Bullet cluster --> difficult for MOND
But the following problems:
Missing baryon mass in clusters (old problem...still there ?)
Distance of galaxies in the bullet cluster not very well known

New result: probing dark halo's at a new mass scale
dark matter structure at the scale of galaxies
Not much problem with missing baryons
Small group, galaxies all the same...

What will be extracted from a large sample of lenses ?

Statistical information on the shape of the dark halo's Potential iso-contours: $dr = f_0(\theta)$

$f_0(\theta) = \sum_n \alpha_n \cos(n\theta + psi_n)$ $\longrightarrow$ Power spectrum

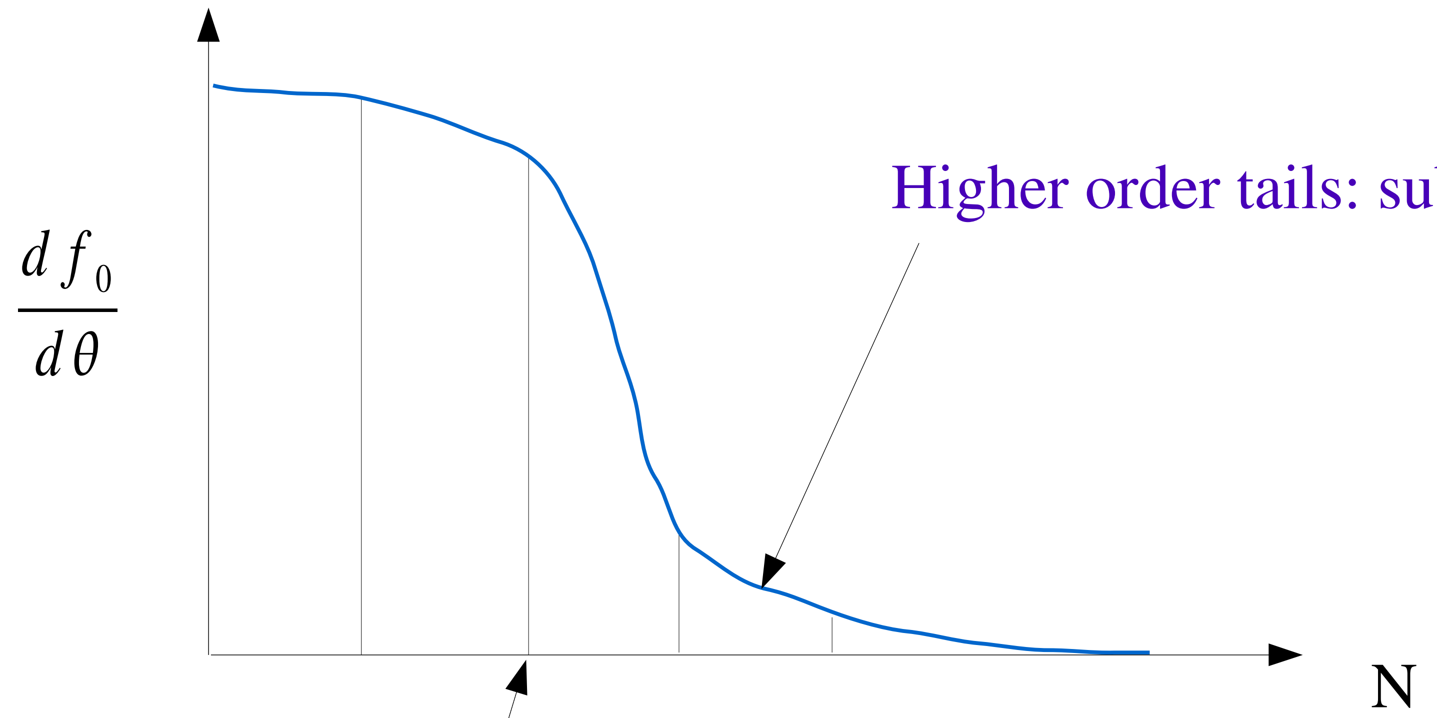


First order Fourier terms: centering

Second order Fourier terms
Ellipticity

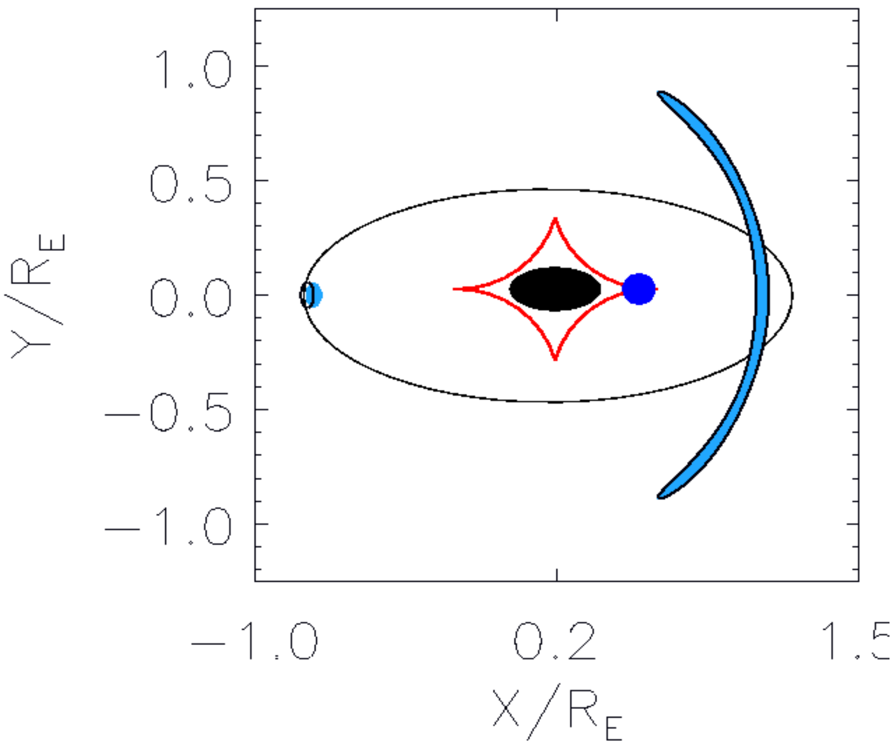
Higher order: substructures ?

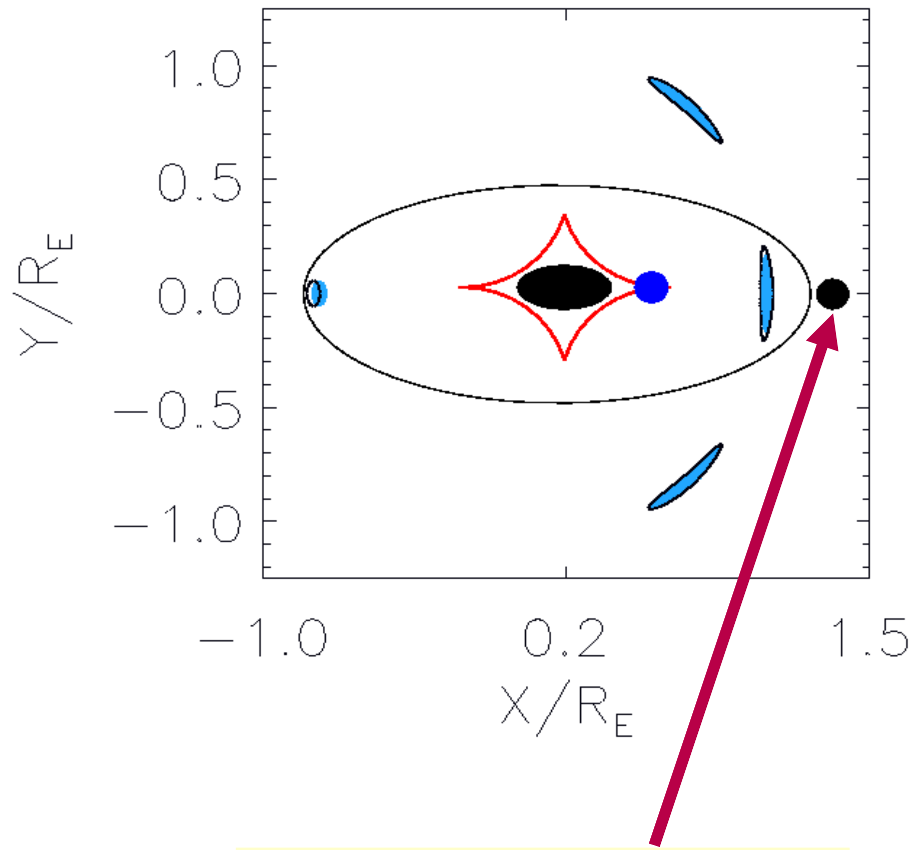
Statistical spectral decomposition of the fields



Order 2 = ellitpticity: most halo's (except mergers)
do not go much beyond order 2

Small substructures can have large effects on arcs

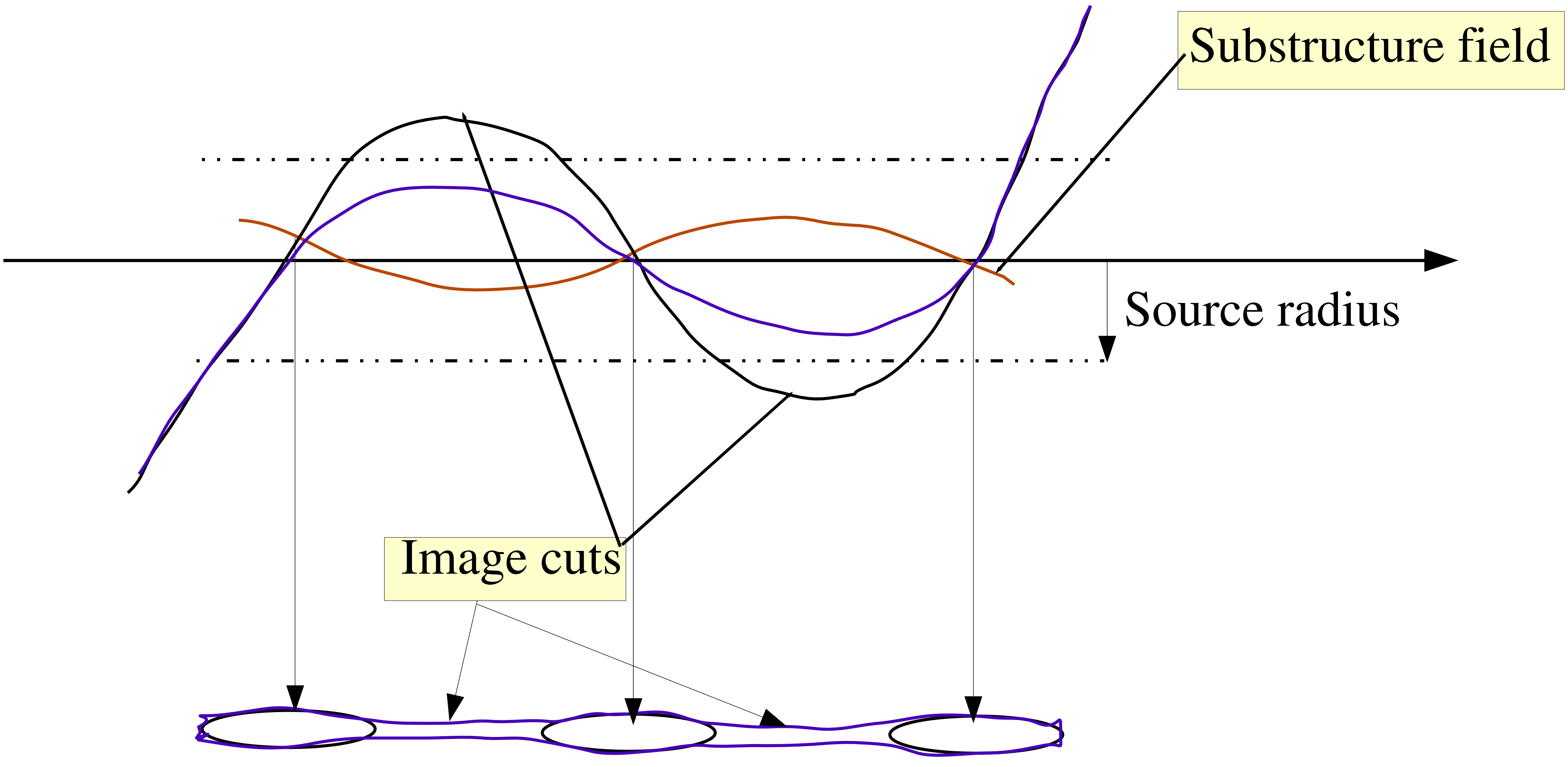




Near Cusp singularity
Elliptical lens

Perturbator mass= 1%

Interpretation

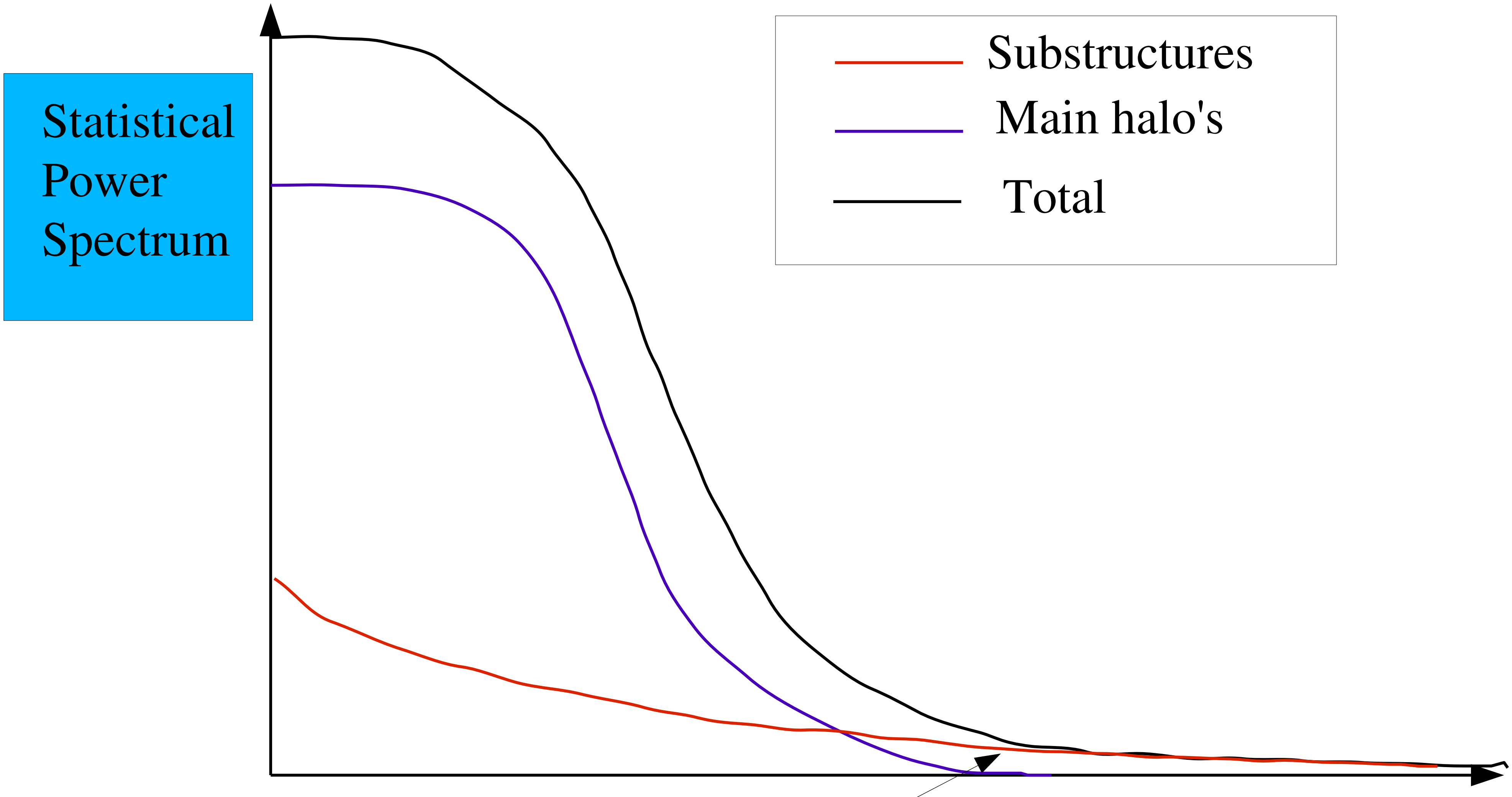


The signature of substructure in the perturbative theory (Alard 2008)

Substructures introduce power law tails in the power spectrum (tend to dominate higher orders)

Amplitude of the power spectrum perturbation identical for the 2 fields

Large slope of the perturbation near the substructure
- strong morphological effects on arcs
- short scale perturbation



Substructures start to dominate the statistical power spectrum at order 4

Finding substructure using the perturbative theory of strong lenses
Will be very much like measuring shear in weak lensing

Know the intrinsic properties of galaxies
Perturbative strong lensing: galaxies power spectrum
Weak lensing: galaxies ellipticities

Estimate residuals
Perturbative strong lensing: Power law tails due to substructures
Weak lensing: Shear

Results:
Perturbative strong lensing: fraction of mass in substructures
Weak lensing: halo mass

Practical implementation of this project: Sebastien Peirani (IAP)
Peirani etal (2008)

Consequences

Comparison of statistical power spectrum to power spectrum
of light distribution: constraints on gravity and dark matter mass
Sebastien Peirani (IAP)

Estimation of substructure mass fraction
If dark substructures: proof of CDM model
If not or weak signal: constraints on the dark matter model
(particle mass, particule type,...warm dark matter ?)
Kev particule mass: De Vega & Sanchez