Le problème des deux corps en relativité générale

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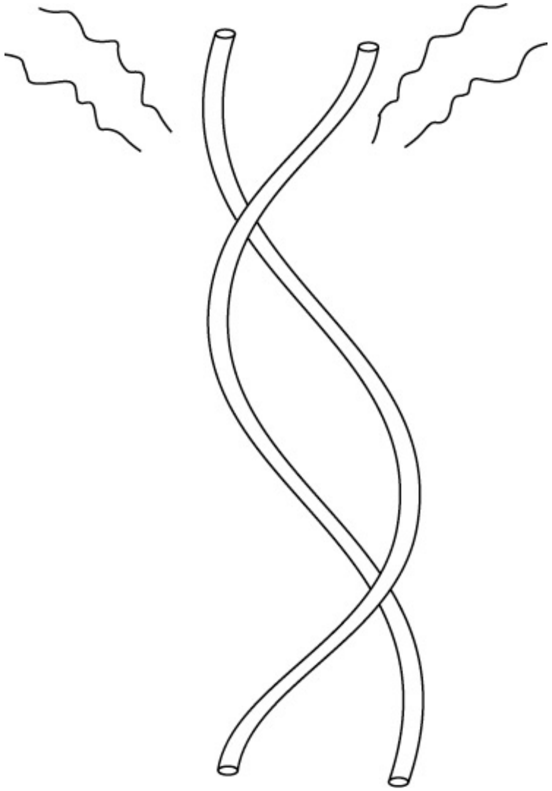
The Problem of Motion in General Relativity

Solve

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

e.g. $$T^{\mu\nu} = (e + p)\, u^\mu u^\nu + p\, g^{\mu\nu}$$



and extract physical results, e.g.

- Lunar laser ranging
- timing of binary pulsars
- gravitational waves emitted by binary black holes

Various issues

Approximation Methods $\left\{ \right.$

- post-Minkowskian (Einstein 1916) $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \ , \ h_{\mu\nu} \ll 1$
- post-Newtonian (Droste 1916) $h_{00} \sim h_{ij} \sim \frac{v^2}{c^2} \ , \ h_{0i} \sim \frac{v^3}{c^3} \ , \ \partial_0 h \sim \frac{v}{c}\partial_i h$
- Matching of asymptotic expansions body zone / near zone / wave zone
- Numerical Relativity

One-chart versus Multi-chart approaches

Coupling between Einstein field equations and equations of motion
(Bianchi $\Rightarrow \nabla^\nu T_{\mu\nu} = 0$)

Strongly self-gravitating bodies : neutron stars or black holes : $h_{\mu\nu}(x) \sim 1$

Skeletonization : $T_{\mu\upsilon} \longrightarrow$ point-masses ? δ-functions in GR

Multipolar Expansion

Need to go to very high orders of approximation

Use a “cocktail”: PM, PN, MPM, MAE, EFT, an. reg., dim. reg., …

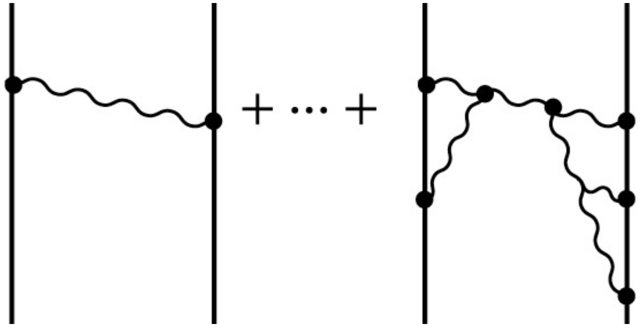
Motion of two point masses

$$S = \int d^D x \frac{R(g)}{16\pi G} - \sum_A \int m_A \sqrt{-g_{\mu\nu}(y_A) dy_A^\mu \, dy_A^\nu}$$

Dimensional continuation : $D = 4 + \varepsilon \; , \; \varepsilon \in \mathbb{C}$

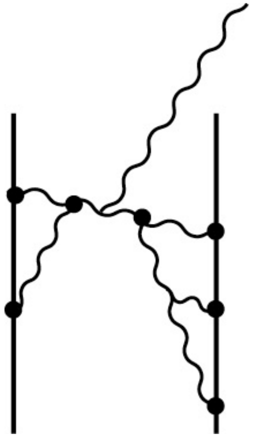
Dynamics : up to 3 loops, i.e. 3 PN

Jaranowski, Schäfer 98
Blanchet, Faye 01
Damour, Jaranowski Schäfer 01
Itoh, Futamase 03
Blanchet, Damour, Esposito-Farèse 04



Radiation : up to 3 PN

Blanchet, Iyer, Joguet, 02,
Blanchet, Damour, Esposito-Farèse, Iyer 04
Blanchet, Faye, Iyer, Sinha 08



2-body Taylor-expanded 3PN Hamiltonian [JS98, DJS00,01]

$$H_{\rm N}(\mathbf{x}_a, \mathbf{p}_a) = \sum_a \frac{\mathbf{p}_a^2}{2\,m_a} - \frac{1}{2}\sum_a \sum_{b\neq a} \frac{G\,m_a\,m_b}{r_{ab}} .$$

$$H_{\rm 1PN}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{1}{8}\frac{(\mathbf{p}_1^2)^2}{m_1^3} + \frac{1}{8}\frac{Gm_1m_2}{r_{12}}\Big[-12\,\frac{\mathbf{p}_1^2}{m_1^2} + 14\,\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + 2\,\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\Big] \quad + \frac{1}{4}\frac{Gm_1m_2}{r_{12}}\frac{G(m_1+m_2)}{r_{12}} + (1 \longleftrightarrow 2),$$

1PN

$$\begin{aligned}
H_{\rm 2PN}(\mathbf{x}_a, \mathbf{p}_a) = {} & \frac{1}{16}\frac{(\mathbf{p}_1^2)^3}{m_1^5} + \frac{1}{8}\frac{Gm_1m_2}{r_{12}}\Big[5\,\frac{(\mathbf{p}_1^2)^2}{m_1^4} - \frac{11}{2}\frac{\mathbf{p}_1^2\,\mathbf{p}_2^2}{m_1^2m_2^2} - \frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} + 5\,\frac{\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} \\
& -6\,\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2} - \frac{3}{2}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\Big] \\
& +\frac{1}{4}\frac{G^2m_1m_2}{r_{12}^2}\Big[m_2\left(10\frac{\mathbf{p}_1^2}{m_1^2} + 19\frac{\mathbf{p}_2^2}{m_2^2}\right) - \frac{1}{2}(m_1+m_2)\frac{27\,(\mathbf{p}_1\cdot\mathbf{p}_2) + 6\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\Big] \\
& -\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\frac{G^2(m_1^2 + 5m_1m_2 + m_2^2)}{r_{12}^2} + (1 \longleftrightarrow 2).
\end{aligned}$$

2PN

$$\begin{aligned}
H_{\rm 3PN}^{\rm reg}(\mathbf{x}_a, \mathbf{p}_a) = {} & -\frac{5}{128}\frac{(\mathbf{p}_1^2)^4}{m_1^7} + \frac{1}{32}\frac{Gm_1m_2}{r_{12}}\Big[-14\,\frac{(\mathbf{p}_1^2)^3}{m_1^6} + 4\,\frac{\big((\mathbf{p}_1\cdot\mathbf{p}_2)^2 + 4\,\mathbf{p}_1^2\,\mathbf{p}_2^2\big)\mathbf{p}_1^2}{m_1^4m_2^2} + \frac{\big(\mathbf{p}_1^2\,\mathbf{p}_2^2 - 2\,(\mathbf{p}_1\cdot\mathbf{p}_2)^2\big)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2^3} \\
& -10\,\frac{\big(\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2 + \mathbf{p}_2^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\big)\mathbf{p}_1^2}{m_1^4m_2^2} + 24\,\frac{\mathbf{p}_1^2\,(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^4m_2^2} + 2\,\frac{\mathbf{p}_1^2\,(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3} \\
& +\frac{\big(7\,\mathbf{p}_1^2\,\mathbf{p}_2^2 - 10\,(\mathbf{p}_1\cdot\mathbf{p}_2)^2\big)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2^3} + 6\,\frac{\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^4m_2^2} \\
& +15\,\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3} - 18\,\frac{\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3} + 5\,\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3}\Big] \\
& +\frac{G^2m_1m_2}{r_{12}^2}\Big[\frac{1}{16}(m_1 - 27m_2)\frac{(\mathbf{p}_1^2)^2}{m_1^4} - \frac{115}{16}m_1\frac{\mathbf{p}_1^2\,(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2} + \frac{1}{48}m_2\frac{25\,(\mathbf{p}_1\cdot\mathbf{p}_2)^2 + 371\,\mathbf{p}_1^2\,\mathbf{p}_2^2}{m_1^2m_2^2} \\
& +\frac{17}{16}\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^3} - \frac{1}{8}m_1\frac{\big(15\,\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_2) + 11\,(\mathbf{p}_1\cdot\mathbf{p}_2)\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)\big)(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1^3m_2} + \frac{5}{12}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^3} \\
& -\frac{3}{2}m_1\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2} + \frac{125}{12}m_2\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2} + \frac{10}{3}m_2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} \\
& -\frac{1}{48}(220m_1 + 193m_2)\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\Big] + \frac{G^3m_1m_2}{r_{12}^3}\Big[-\frac{1}{48}\left(466\,m_1^2 + \left(473 - \frac{3}{4}\pi^2\right)m_1m_2 + 150\,m_2^2\right)\frac{\mathbf{p}_1^2}{m_1^2} \\
& +\frac{1}{16}\left(77(m_1^2 + m_2^2) + \left(143 - \frac{1}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + \frac{1}{16}\left(61\,m_1^2 - \left(43 + \frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2} \\
& +\frac{1}{16}\left(21(m_1^2 + m_2^2) + \left(119 + \frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\Big] \\
& +\frac{1}{8}\frac{G^4m_1m_2^3}{r_{12}^4}\left[\left(\frac{227}{3} - \frac{21}{4}\pi^2\right)m_1 + m_2\right] + (1 \longleftrightarrow 2). \qquad (12)
\end{aligned}$$

3PN

Taylor-expanded 3PN waveform

Blanchet,Iyer, Joguet 02, Blanchet, Damour, Esposito-Farese, Iyer 04, Kidder 07, Blanchet et al. 08

$$
\begin{aligned}
h^{22} = & -8\sqrt{\frac{\pi}{5}}\frac{G\nu m}{c^2R}e^{-2i\phi}x\Big\{1 - x\Big(\frac{107}{42} - \frac{55}{42}\nu\Big) + x^{3/2}\Big[2\pi + 6i\ln\Big(\frac{x}{x_0}\Big)\Big] - x^2\Big(\frac{2173}{1512} + \frac{1069}{216}\nu - \frac{2047}{1512}\nu^2\Big) \\
& - x^{5/2}\Big[\Big(\frac{107}{21} - \frac{34}{21}\nu\Big)\pi + 24i\nu + \Big(\frac{107i}{7} - \frac{34i}{7}\nu\Big)\ln\Big(\frac{x}{x_0}\Big)\Big] \\
& + x^3\Big[\frac{27\,027\,409}{646\,800} - \frac{856}{105}\gamma_E + \frac{2}{3}\pi^2 - \frac{1712}{105}\ln 2 - \frac{428}{105}\ln x \\
& - 18\Big[\ln\Big(\frac{x}{x_0}\Big)\Big]^2 - \Big(\frac{278\,185}{33\,264} - \frac{41}{96}\pi^2\Big)\nu - \frac{20\,261}{2772}\nu^2 + \frac{114\,635}{99\,792}\nu^3 + \frac{428i}{105}\pi + 12i\pi\ln\Big(\frac{x}{x_0}\Big)\Big] + O(\epsilon^{7/2})\Big\},
\end{aligned}
$$

$$x = (M\Omega)^{2/3} \sim v^2/c^2$$

$$
\begin{aligned}
M &= m_1 + m_2 \\
\nu &= m_1 m_2/(m_1 + m_2)^2
\end{aligned}
$$

Renewed importance of 2-body problem

- Gravitational wave (GW) signal emitted by binary black hole coalescences : a prime target for LIGO/Virgo/GEO

- GW signal emitted by binary neutron stars : target for advanced LIGO….

BUT

- Breakdown of analytical approach in such strong-field situations ? expansion parameter $x \sim \frac{v^2}{c^2} \sim \mathcal{O}(1)$ during coalescence ! ?

- Give up analytical approach, and use only Numerical Relativity ?

Binary black hole coalescence

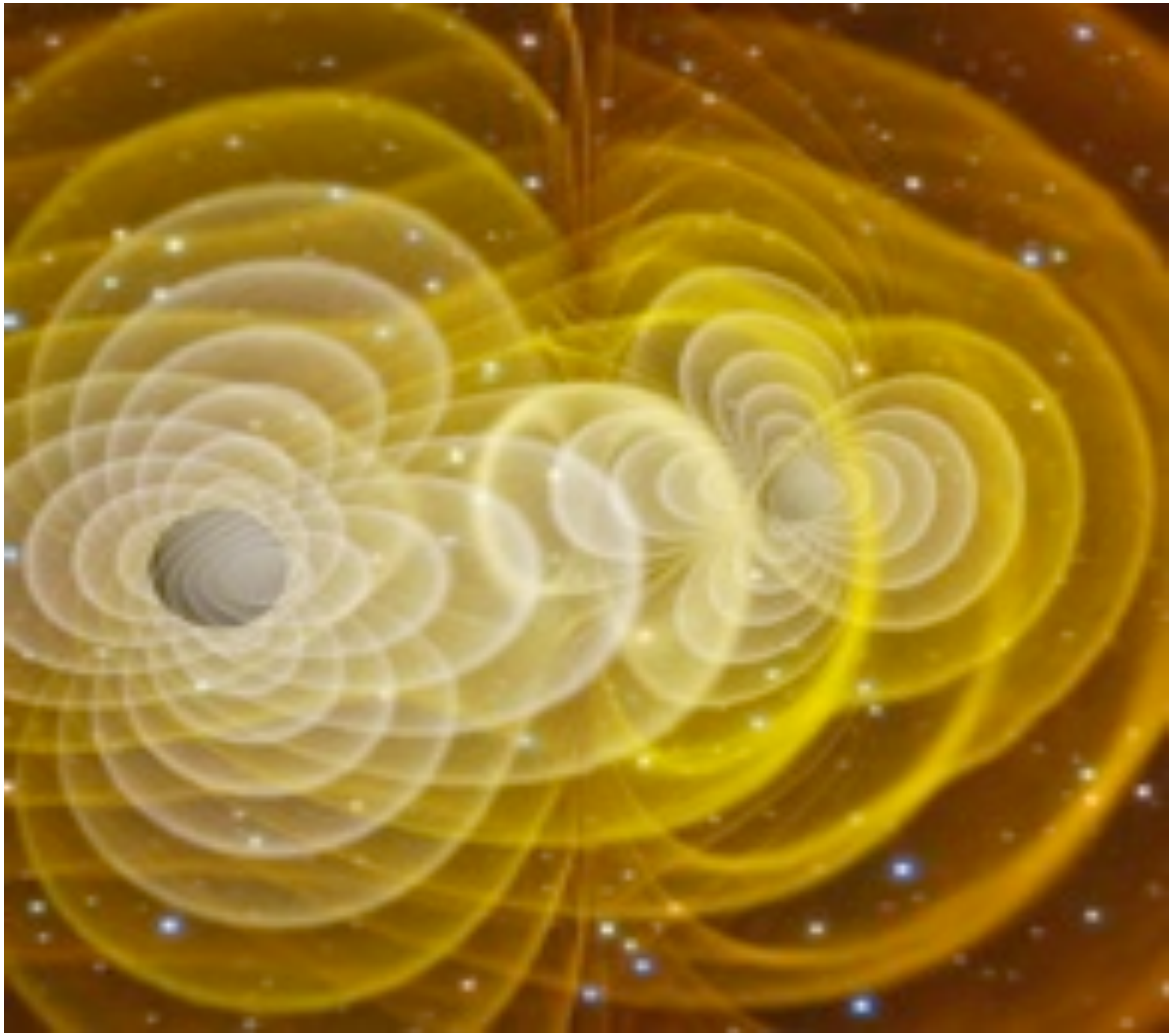
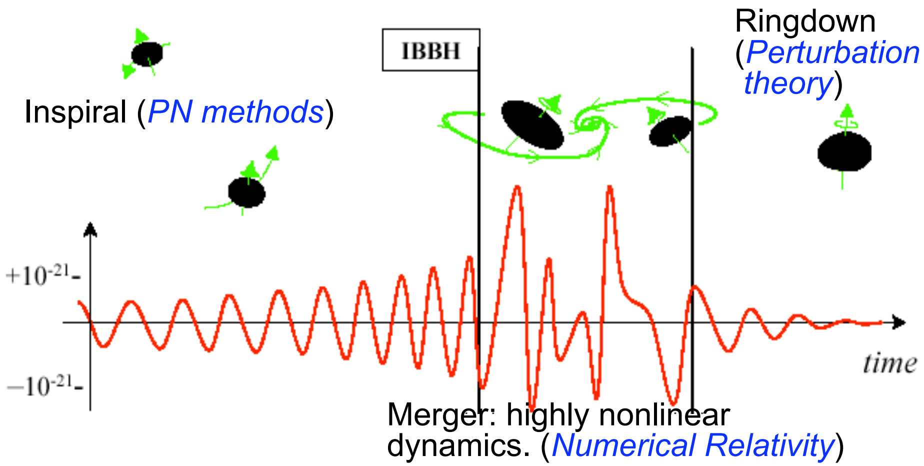


Image: NASA/GSFC

Templates for GWs from BBH coalescence

(Brady, Craighton, Thorne 1998)

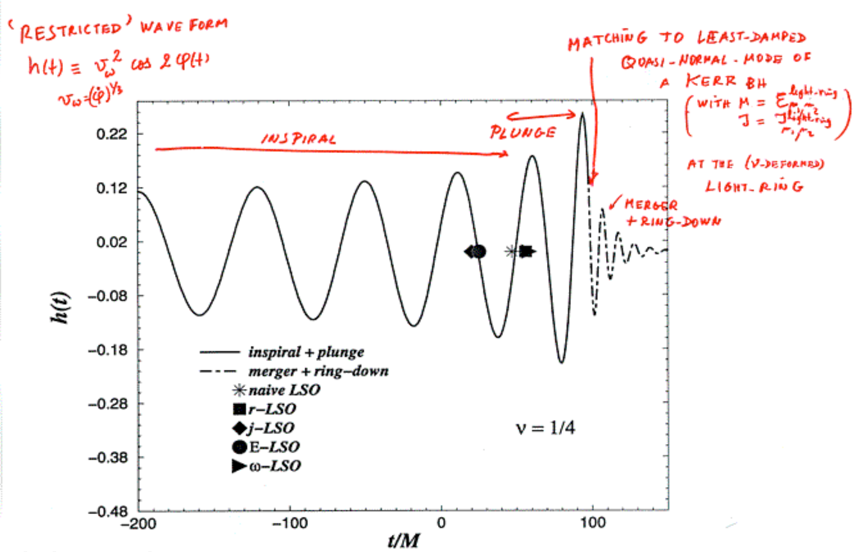


Inspiral (*PN methods*)

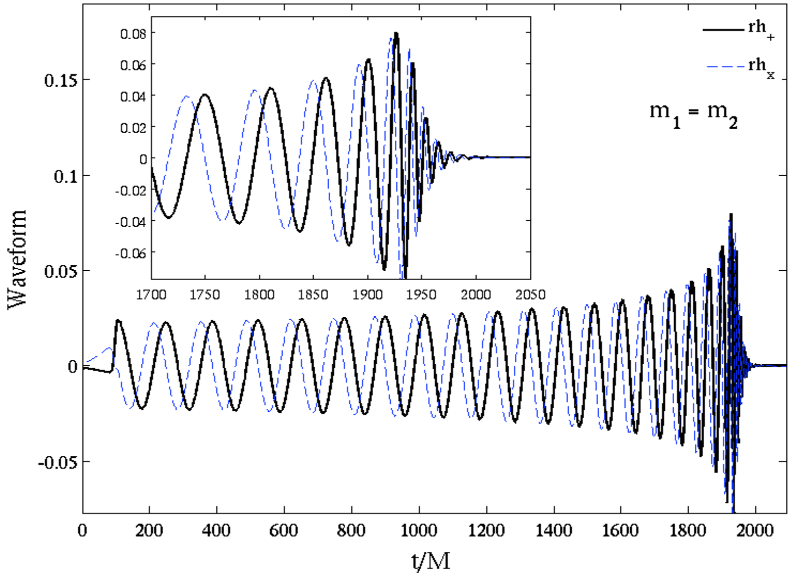
Ringdown (*Perturbation theory*)

Merger: highly nonlinear dynamics. (*Numerical Relativity*)

(Buonanno & Damour 2000)



Numerical Relativity, the 2005 breakthrough:
Pretorius, Campanelli et al., Baker et al. …



An improved analytical approach

EFFECTIVE ONE BODY (EOB)

approach to the two-body problem

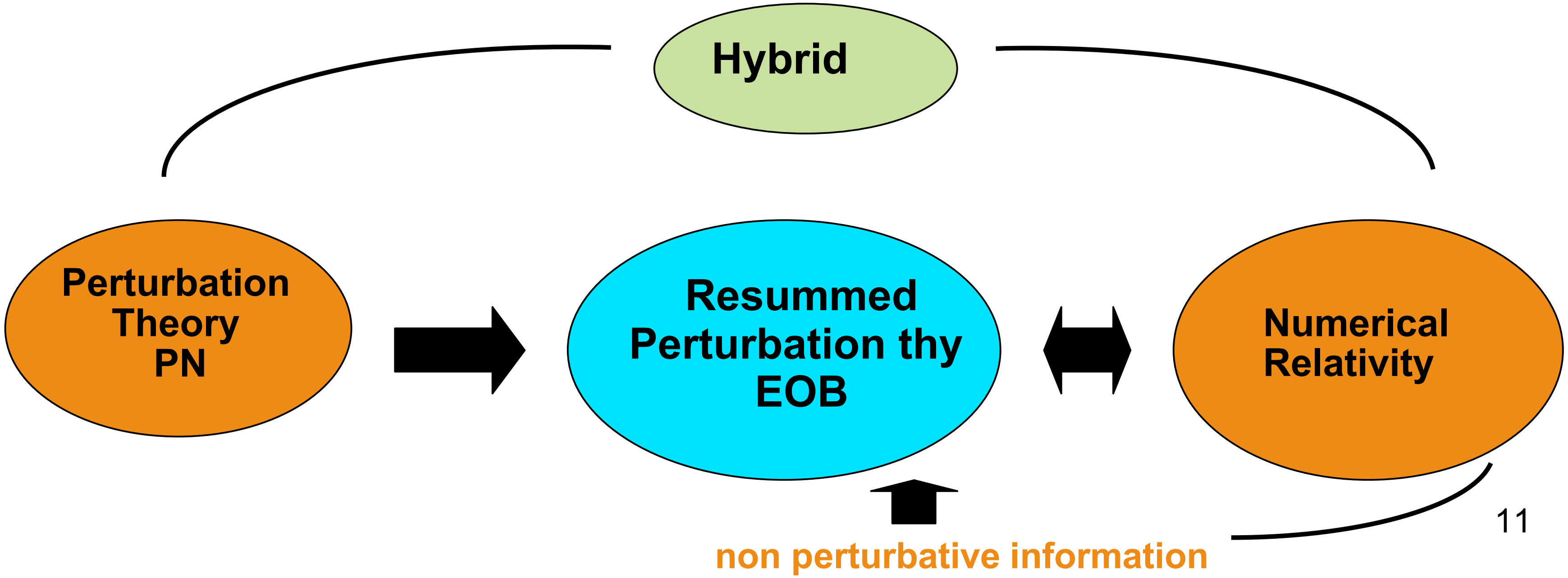
Buonanno,Damour 99	(2 PN Hamiltonian)
Buonanno,Damour 00	(Rad.Reac. full waveform)
Damour, Jaranowski,Schäfer 00	(3 PN Hamiltonian)
Damour, 01	(spin)
Damour, Nagar 07, Damour, Iyer, Nagar 08	(factorized waveform)

Importance of an analytical formalism

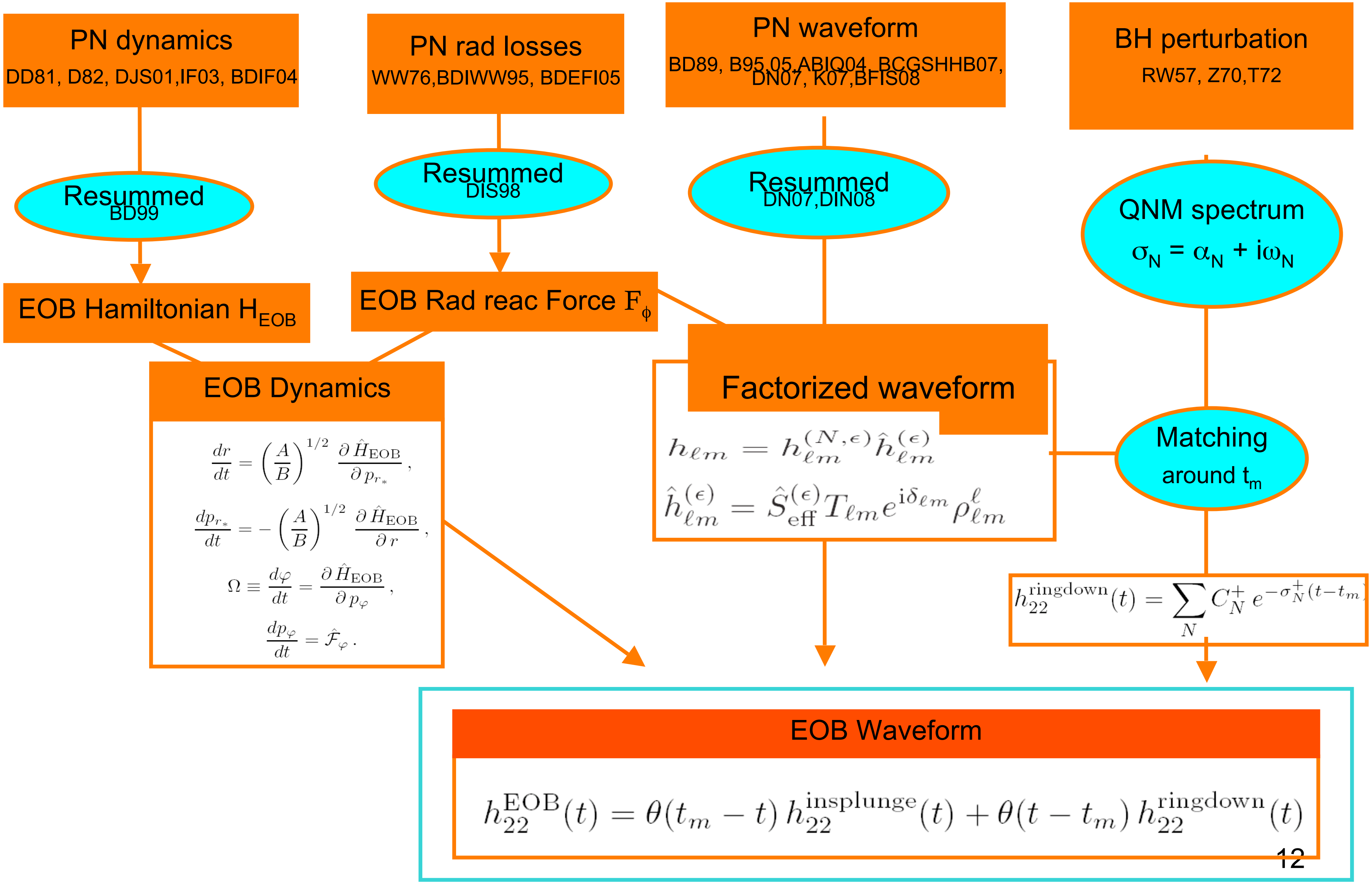
- **Theoretical:** physical understanding of the coalescence process, especially in complicated situations (arbitrary spins)

- **Practical:** need many thousands of accurate GW templates for detection & data analysis;
  need some "analytical" representation of waveform templates as $f(m_1,m_2,\mathbf{S}_1,\mathbf{S}_2)$

- Solution: **synergy between analytical & numerical relativity**



Structure of EOB formalism



PN dynamics
DD81, D82, DJS01,IF03, BDIF04

Resummed
BD99

EOB Hamiltonian H_{EOB}

PN rad losses
WW76,BDIWW95, BDEFI05

Resummed
DIS98

EOB Rad reac Force F_{ϕ}

PN waveform
BD89, B95,05,ABIQ04, BCGSHHB07,
DN07, K07,BFIS08

Resummed
DN07,DIN08

BH perturbation
RW57, Z70,T72

QNM spectrum
$\sigma_N = \alpha_N + i\omega_N$

EOB Dynamics

$$\frac{dr}{dt} = \left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\rm EOB}}{\partial p_{r_*}},$$

$$\frac{dp_{r_*}}{dt} = -\left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\rm EOB}}{\partial r},$$

$$\Omega \equiv \frac{d\varphi}{dt} = \frac{\partial \hat{H}_{\rm EOB}}{\partial p_{\varphi}},$$

$$\frac{dp_{\varphi}}{dt} = \hat{\mathcal{F}}_{\varphi}\,.$$

Factorized waveform

$$h_{\ell m} = h^{(N,\epsilon)}_{\ell m}\,\hat{h}^{(\epsilon)}_{\ell m}$$

$$\hat{h}^{(\epsilon)}_{\ell m} = \hat{S}^{(\epsilon)}_{\rm eff}\,T_{\ell m}\,e^{{\rm i}\delta_{\ell m}}\,\rho^{\ell}_{\ell m}$$

Matching
around t_m

$$h^{\rm ringdown}_{22}(t) = \sum_N C^+_N\, e^{-\sigma^+_N(t-t_m)}$$

EOB Waveform

$$h^{\rm EOB}_{22}(t) = \theta(t_m - t)\, h^{\rm insplunge}_{22}(t) + \theta(t - t_m)\, h^{\rm ringdown}_{22}(t)$$

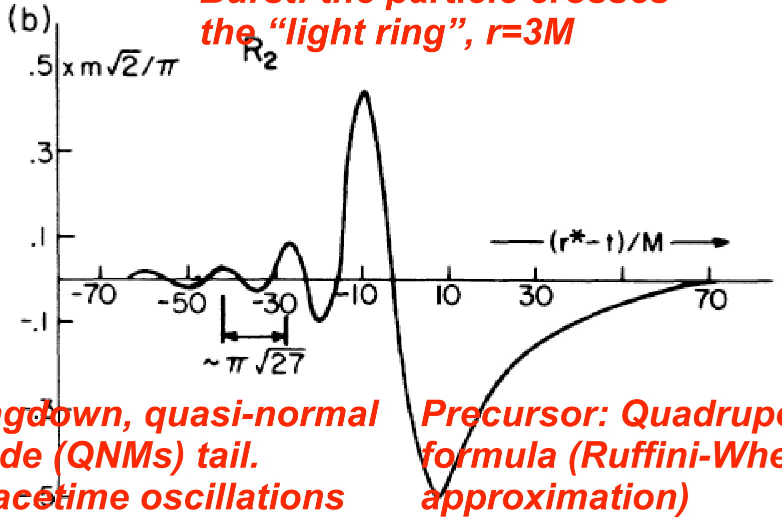
Historical roots of EOB

- H_{EOB} : QED positronium states [Brezin, Itzykson, Zinn-Justin 1970]
  "Quantum" Hamiltonian $H(I_a)$ [Damour-Schäfer 1988]

- Padé resummation [Padé1892]
  - F_ϕ [DIS1998]
  - A(r) [DJS00]
  - Factorized waveform [DN07]

- h(t) : [Davis, Ruffini, Tiomno 1972]
  CLAP [Price-Pullin 1994]

Discovery of the structure:
Precursor (plunge)-Burst (merger)-Ringdown



Burst: the particle crosses the "light ring", r=3M

Ringdown, quasi-normal mode (QNMs) tail. Spacetime oscillations

Precursor: Quadrupole formula (Ruffini-Wheeler approximation)

Some key references

PN

Wagoner & Will 76
Damour & Deruelle 81,82;
Blanchet & Damour 86
Damour & Schafer 88
Blanchet & Damour 89;
Blanchet, Damour Iyer, Will, Wiseman 95
Blanchet 95
Jaranowski & Schafer 98
Damour, Jaranowski, Schafer 01
Blanchet, Damour, Esposito-Farese & Iyer 05
Kidder 07
Blanchet, Faye, Iyer & Sinha, 08

NR

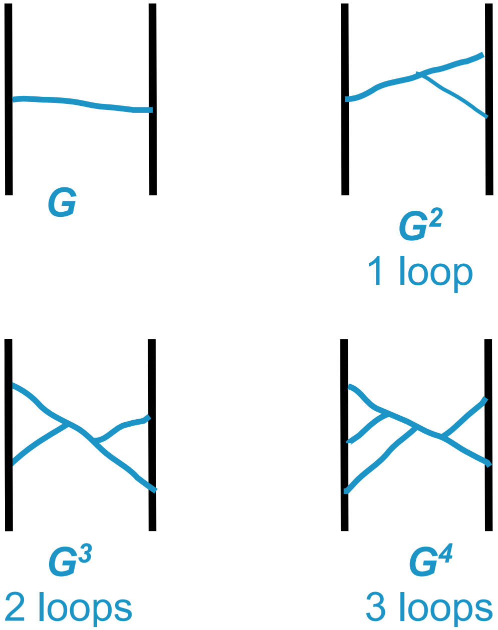
Brandt & Brugmann 97
Baker, Brugmann, Campanelli, Lousto & Takahashi 01
Baker, Campanelli, Lousto & Takahashi 02
Pretorius 05
Baker et al. 05
Campanelli et al. 05
Gonzalez et al. 06
Koppitz et al. 07
Pollney et al. 07
Boyle et al. 07
Scheel et al. 08

EOB

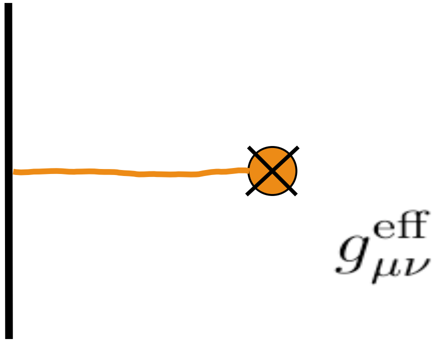
Buonanno & Damour 99, 00
Damour 01
Damour Jaranowski & Schafer 00
Buonanno et al. 06-09
Damour & Nagar 07-09
Damour, Iyer & Nagar 08

Real dynamics versus Effective dynamics

Real dynamics



Effective dynamics



$$S = -\int \mu ds + \dots$$

$$H = H_0 + \left(G H_1 + \frac{G^2}{c^2} H_2 + \frac{G^3}{c^4} H_3 + \frac{G^4}{c^6} H_4 \right) \left(1 + \frac{1}{c^2} + \dots \right)$$

Effective metric

$$ds^2_{\text{eff}} = -A(r)dt^2 + B(r)dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right)$$

Two-body/EOB "correspondence": think quantum-mechanically (Wheeler)

Real 2-body system (m_1, m_2) (in the c.o.m. frame)

an effective particle of mass μ in some effective metric $g_{\mu\nu}^{eff}(M)$

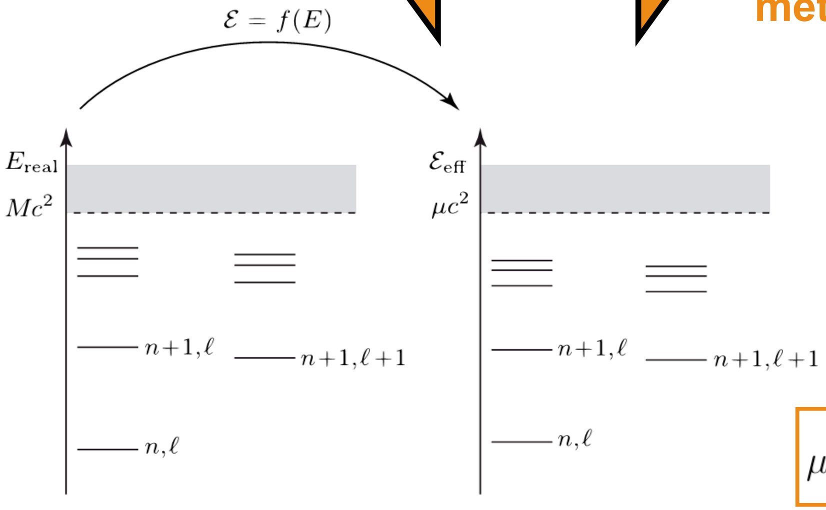


Figure 1: Sketch of the correspondence between the quantized energy levels of the real and effective conservative dynamics. n denotes the 'principal quantum

$$\mu^2 + g_{\rm eff}^{\mu\nu} \frac{\partial S_{\rm eff}}{\partial x^\mu} \frac{\partial S_{\rm eff}}{\partial x^\nu} + \mathcal{O}(p^4) = 0$$

Sommerfeld "Old Quantum Mechanics":

$$J = \ell\hbar = \frac{1}{2\pi} \oint p_\varphi \, d\varphi$$
$$N = n\hbar = I_r + J$$
$$I_r = \frac{1}{2\pi} \oint p_r \, dr$$

$H^{classical}(q,p)$ → $H^{classical}(I_a)$ → $E^{\rm quantum}(I_a = n_a h) = f^{-1}\left[\mathcal{E}_{\rm eff}^{\rm quantum}\left(I_a^{\rm eff} = n_a h\right)\right]$

The 3PN EOB Hamiltonian

Real 2-body system ***(m_1, m_2)***
(in the c.o.m. frame)

1:1 map

an effective particle of mass ***$\mu = m_1 m_2/(m_1+m_2)$*** **in some effective metric** ***$g_{\mu\nu}^{eff}(M)$***

Simple energy map

$$\mathcal{E}_{eff} = \frac{s - m_1^2 - m_2^2}{2M}$$

$$s = E_{\rm real}^2$$

$$H_{\rm EOB} = M\sqrt{1 + 2\nu\left(\hat{H}_{\rm eff} - 1\right)}$$

$$\begin{aligned} M &= m_1 + m_2 \\ \nu &= m_1 m_2/(m_1 + m_2)^2 \end{aligned}$$

Simple effective Hamiltonian

$$\hat{H}_{\rm eff} \equiv \sqrt{p_{r_*}^2 + A\left(1 + \frac{p_\varphi^2}{r^2} + z_3\frac{p_{r_*}^4}{r^2}\right)}.$$

crucial EOB "radial potential" A(r)

$$p_{r_*} = \left(\frac{A}{B}\right)^{1/2} p_r$$

Explicit form of the effective metric

The effective metric $g_{\mu\nu}^{eff}(M)$ at 3PN

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

where the coefficients are a ν-dependent “deformation” of the Schwarzschild ones:

$$\boxed{A_{3\mathrm{PN}}(R) = 1 - 2u + 2\nu\, u^3 + a_4 \nu\, u^4}$$

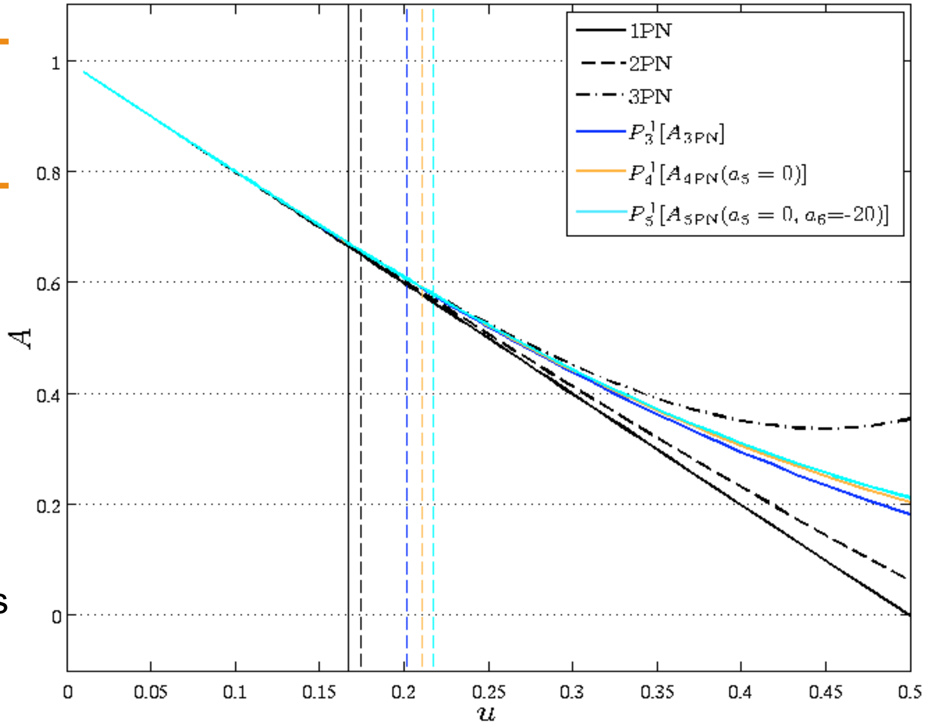
$$a_4 = \frac{94}{3} - \frac{41}{32}\pi^2 \simeq 18.6879027$$

$$(A(R)B(R))_{3\mathrm{PN}} = 1 - 6\nu u^2 + 2(3\nu - 26)\nu u^3$$

u = 1/r

$$A(u; a5, a6, \nu) = P_5^1\left[A_{3\mathrm{PN}}(u) + \nu a_5 u^5 + \nu a_6 u^6\right]$$

- Compact representation of PN dynamics
- Bad behaviour at 3PN. Use Padé resummation of A(r) to have an effective horizon.
- Impose [by continuity with the ν=0 case] that A(r) has a simple zero [at r≈2].
- The **a_5** and **a_6** constants parametrize (yet) uncalculated **4PN** corrections and **5PN** corrections



2-body Taylor-expanded 3PN Hamiltonian [JS98, DJS00,01]

$$H_{\mathrm{N}}(\mathbf{x}_a,\mathbf{p}_a)=\sum_a \frac{\mathbf{p}_a^2}{2\,m_a}-\frac{1}{2}\sum_a\sum_{b\neq a}\frac{G\,m_a\,m_b}{r_{ab}}\,.$$

$$H_{1\mathrm{PN}}(\mathbf{x}_a,\mathbf{p}_a)=-\frac{1}{8}\frac{(\mathbf{p}_1^2)^2}{m_1^3}+\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left[-12\,\frac{\mathbf{p}_1^2}{m_1^2}+14\,\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2}+2\,\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right]\quad+\frac{1}{4}\frac{Gm_1m_2}{r_{12}}\frac{G(m_1+m_2)}{r_{12}}+\big(1\longleftrightarrow 2\big),$$

1PN

$$\begin{aligned}H_{2\mathrm{PN}}(\mathbf{x}_a,\mathbf{p}_a)=&\;\frac{1}{16}\frac{(\mathbf{p}_1^2)^3}{m_1^5}+\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left[5\,\frac{(\mathbf{p}_1^2)^2}{m_1^4}-\frac{11}{2}\,\frac{\mathbf{p}_1^2\,\mathbf{p}_2^2}{m_1^2m_2^2}-\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}+5\,\frac{\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right.\\&\left.-6\,\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2}-\frac{3}{2}\,\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right]\\&+\frac{1}{4}\frac{G^2m_1m_2}{r_{12}^2}\left[m_2\left(10\frac{\mathbf{p}_1^2}{m_1^2}+19\frac{\mathbf{p}_2^2}{m_2^2}\right)-\frac{1}{2}(m_1+m_2)\frac{27\,(\mathbf{p}_1\cdot\mathbf{p}_2)+6\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right]\\&-\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\frac{G^2(m_1^2+5m_1m_2+m_2^2)}{r_{12}^2}+\big(1\longleftrightarrow 2\big).\end{aligned}$$

2PN

$$\begin{aligned}H^{\mathrm{reg}}_{3\mathrm{PN}}(\mathbf{x}_a,\mathbf{p}_a)=&-\frac{5}{128}\frac{(\mathbf{p}_1^2)^4}{m_1^7}+\frac{1}{32}\frac{Gm_1m_2}{r_{12}}\left[-14\,\frac{(\mathbf{p}_1^2)^3}{m_1^6}+4\,\frac{\big((\mathbf{p}_1\cdot\mathbf{p}_2)^2+4\,\mathbf{p}_1^2\,\mathbf{p}_2^2\big)\mathbf{p}_1^2}{m_1^4m_2^2}+\frac{\big(\mathbf{p}_1^2\,\mathbf{p}_2^2-2\,(\mathbf{p}_1\cdot\mathbf{p}_2)^2\big)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2^3}\right.\\&-10\,\frac{\big(\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2+\mathbf{p}_2^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\big)\mathbf{p}_1^2}{m_1^4m_2^2}+24\,\frac{\mathbf{p}_1^2\,(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^4m_2^2}+2\,\frac{\mathbf{p}_1^2\,(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3}\\&+\frac{\big(7\,\mathbf{p}_1^2\,\mathbf{p}_2^2-10\,(\mathbf{p}_1\cdot\mathbf{p}_2)^2\big)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2^3}+6\,\frac{\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^4m_2^2}\\&\left.+15\,\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3}-18\,\frac{\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3}+5\,\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3}\right]\\&+\frac{G^2m_1m_2}{r_{12}^2}\left[\frac{1}{16}(m_1-27m_2)\frac{(\mathbf{p}_1^2)^2}{m_1^4}-\frac{115}{16}m_1\frac{\mathbf{p}_1^2\,(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2}+\frac{1}{48}m_2\frac{25\,(\mathbf{p}_1\cdot\mathbf{p}_2)^2+371\,\mathbf{p}_1^2\,\mathbf{p}_2^2}{m_1^2m_2^2}\right.\\&+\frac{17}{16}\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^3}-\frac{1}{8}m_1\frac{\big(15\,\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_2)+11\,(\mathbf{p}_1\cdot\mathbf{p}_2)\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)\big)(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1^3m_2}+\frac{5}{12}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^3}\\&-\frac{3}{2}m_1\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2}+\frac{125}{12}m_2\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2}+\frac{10}{3}m_2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\\&\left.-\frac{1}{48}(220m_1+193m_2)\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right]+\frac{G^3m_1m_2}{r_{12}^3}\left[-\frac{1}{48}\left(466\,m_1^2+\left(473-\frac{3}{4}\pi^2\right)m_1m_2+150\,m_2^2\right)\frac{\mathbf{p}_1^2}{m_1^2}\right.\\&+\frac{1}{16}\left(77(m_1^2+m_2^2)+\left(143-\frac{1}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2}+\frac{1}{16}\left(61\,m_1^2-\left(43+\frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2}\\&\left.+\frac{1}{16}\left(21(m_1^2+m_2^2)+\left(119+\frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right]\\&+\frac{1}{8}\frac{G^4m_1m_2^3}{r_{12}^4}\left[\left(\frac{227}{3}-\frac{21}{4}\pi^2\right)m_1+m_2\right]+\big(1\longleftrightarrow 2\big).\end{aligned}\tag{12}$$

3PN

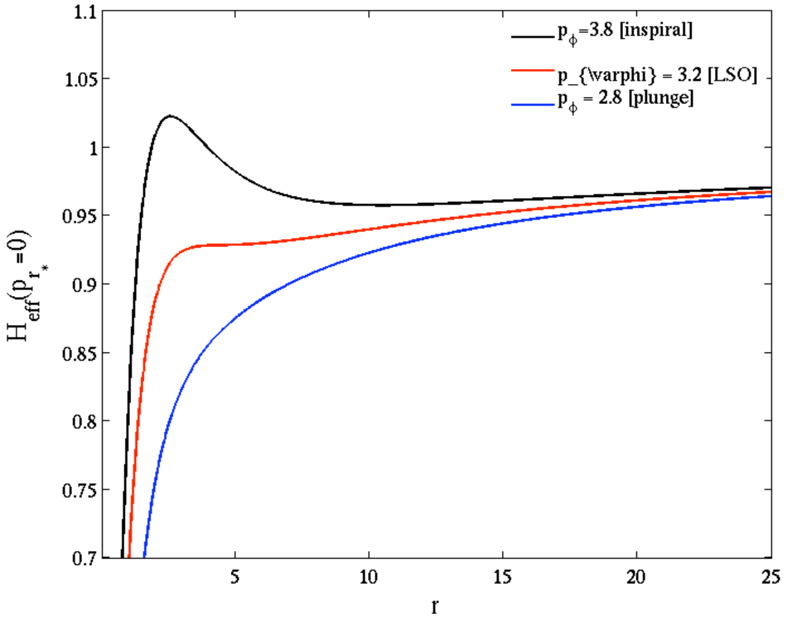
Hamilton's equation + radiation reaction

$$\frac{dr}{dt} = \left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\rm EOB}}{\partial p_{r_*}} \,,$$

$$\frac{dp_{r_*}}{dt} = -\left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\rm EOB}}{\partial r} \,,$$

$$\Omega \equiv \frac{d\varphi}{dt} = \frac{\partial \hat{H}_{\rm EOB}}{\partial p_\varphi} \,,$$

$$\boxed{\frac{dp_\varphi}{dt} = \hat{\mathcal{F}}_\varphi \,.}$$

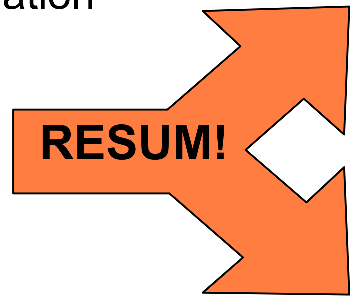


The system must lose mechanical angular momentum

Use PN-expanded result for ***GW angular momentum flux*** as a starting point.
Needs resummation to have a better behavior during late-inspiral and plunge.

PN calculations are done in the circular approximation

$$\hat{\mathcal{F}}_\varphi^{\rm Taylor} = -\frac{32}{5}\,\nu\,\Omega^5\, r_\omega^4\, \hat{F}^{\rm Taylor}(v_\varphi)$$



Parameter-dependent

EOB 1.* **[DIS 1998, DN07]**

Parameter -free:

EOB 2.0 **[DIN 2008, DN09]**

Taylor-expanded 3PN waveform

Blanchet,Iyer, Joguet 02, Blanchet, Damour, Esposito-Farese, Iyer 04, Kidder 07, Blanchet et al. 08

$$
\begin{aligned}
h^{22} = {} & -8\sqrt{\frac{\pi}{5}}\frac{G\nu m}{c^2R}e^{-2i\phi}x\Big\{1 - x\Big(\frac{107}{42} - \frac{55}{42}\nu\Big) + x^{3/2}\Big[2\pi + 6i\ln\Big(\frac{x}{x_0}\Big)\Big] - x^2\Big(\frac{2173}{1512} + \frac{1069}{216}\nu - \frac{2047}{1512}\nu^2\Big) \\
& - x^{5/2}\Big[\Big(\frac{107}{21} - \frac{34}{21}\nu\Big)\pi + 24i\nu + \Big(\frac{107i}{7} - \frac{34i}{7}\nu\Big)\ln\Big(\frac{x}{x_0}\Big)\Big] \\
& + x^3\Big[\frac{27\,027\,409}{646\,800} - \frac{856}{105}\gamma_E + \frac{2}{3}\pi^2 - \frac{1712}{105}\ln 2 - \frac{428}{105}\ln x \\
& - 18\Big[\ln\Big(\frac{x}{x_0}\Big)\Big]^2 - \Big(\frac{278\,185}{33\,264} - \frac{41}{96}\pi^2\Big)\nu - \frac{20\,261}{2772}\nu^2 + \frac{114\,635}{99\,792}\nu^3 + \frac{428i}{105}\pi + 12i\pi\ln\Big(\frac{x}{x_0}\Big)\Big] + O(\epsilon^{7/2})\Big\},
\end{aligned}
$$

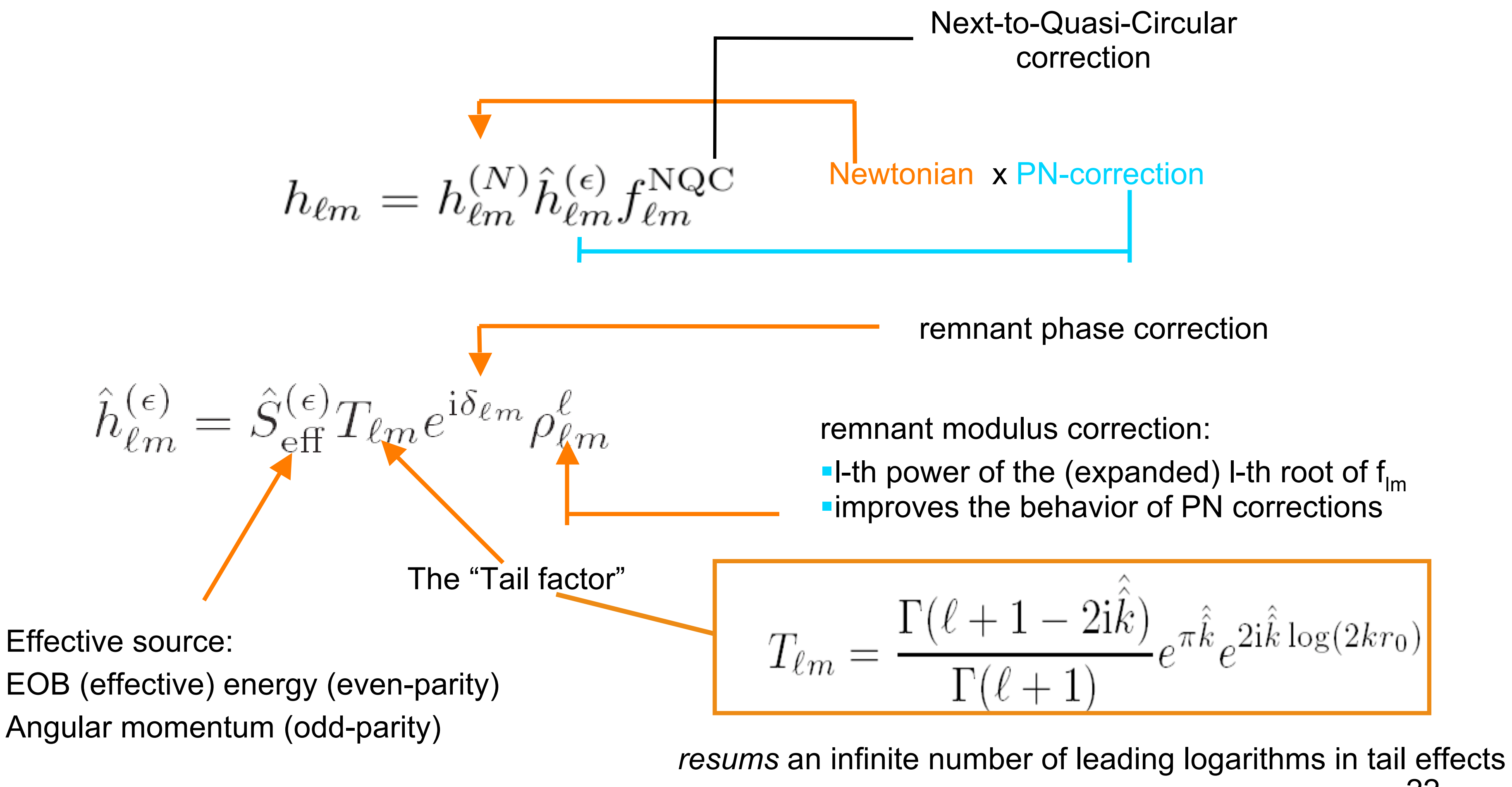
$$x = (M\Omega)^{2/3} \sim v^2/c^2$$

$$M = m_1 + m_2$$

$$\nu = m_1 m_2/(m_1 + m_2)^2$$

EOB 2.0: new resummation procedures (DN07, DIN 2008)

- Resummation of the waveform **multipole by multipole**
- **Factorized** waveform for any (l,m) at the highest available PN order (start from PN results of Blanchet et al.)



Next-to-Quasi-Circular correction

$$h_{\ell m} = h^{(N)}_{\ell m}\,\hat{h}^{(\epsilon)}_{\ell m}\,f^{\rm NQC}_{\ell m}$$

Newtonian x PN-correction

remnant phase correction

$$\hat{h}^{(\epsilon)}_{\ell m} = \hat{S}^{(\epsilon)}_{\rm eff}\,T_{\ell m}\,e^{{\rm i}\delta_{\ell m}}\,\rho^{\ell}_{\ell m}$$

remnant modulus correction:

- l-th power of the (expanded) l-th root of f_{lm}
- improves the behavior of PN corrections

The “Tail factor”

Effective source:

EOB (effective) energy (even-parity)

Angular momentum (odd-parity)

$$T_{\ell m} = \frac{\Gamma(\ell+1-2{\rm i}\hat{\hat{k}})}{\Gamma(\ell+1)}\,e^{\pi\hat{\hat{k}}}\,e^{2{\rm i}\hat{\hat{k}}\log(2kr_0)}$$

resums an infinite number of leading logarithms in tail effects

Radiation reaction: parameter-free resummation

$$\mathcal{F}_\varphi \equiv -\frac{1}{8\pi\,\Omega}\sum_{\ell=2}^{\ell_{\rm max}}\sum_{m=1}^{\ell}(m\,\Omega)^2\,|R\,h_{\ell m}^{(\epsilon)}|^2$$

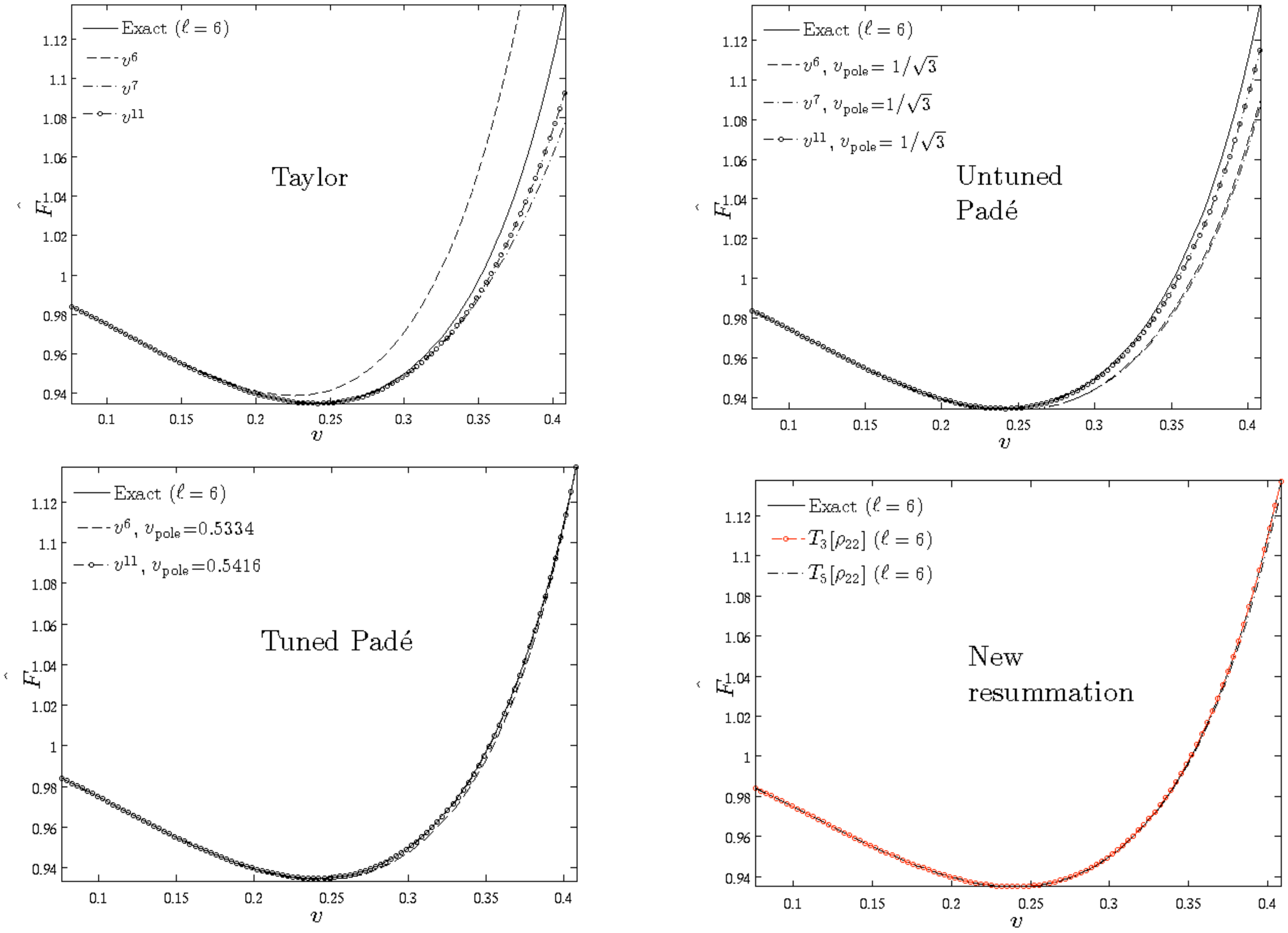
$$h_{\ell m} = h_{\ell m}^{(N)}\hat{h}_{\ell m}^{(\epsilon)}f_{\ell m}^{\rm NQC}$$

$$\hat{h}_{\ell m}^{(\epsilon)} = \hat{S}_{\rm eff}^{(\epsilon)}T_{\ell m}e^{{\rm i}\delta_{\ell m}}\rho_{\ell m}^{\ell}$$

$$\begin{aligned}\rho_{22}(x;\nu) = 1 &+ \left(\frac{55\nu}{84}-\frac{43}{42}\right)x + \left(\frac{19583\nu^2}{42336}-\frac{33025\nu}{21168}-\frac{20555}{10584}\right)x^2\\ &+ \left(\frac{10620745\nu^3}{39118464}-\frac{6292061\nu^2}{3259872}+\frac{41\pi^2\nu}{192}-\frac{48993925\nu}{9779616}-\frac{428}{105}\text{eulerlog}_2(x)+\frac{1556919113}{122245200}\right)x^3\\ &+ \left(\frac{9202}{2205}\text{eulerlog}_2(x)-\frac{387216563023}{160190110080}\right)x^4 + \left(\frac{439877}{55566}\text{eulerlog}_2(x)-\frac{16094530514677}{533967033600}\right)x^5 + \mathcal{O}(x^6),\end{aligned}$$

- Different possible representations of the residual amplitude correction [Padé]
- The “adiabatic” EOB parameters (***a_5***, ***a_6***) propagate in radiation reaction via the effective source.

Test-mass limit (ν=0): circular orbits



Parameter free resummation technique!

EOB 2.0: Next-to-Quasi-Circular correction: EOB ∪ NR

Next-to quasi-circular correction to the *l=m=2 amplitude*

$$\boxed{f_{22}^{\mathrm{NQC}}(a_1, a_2) = 1 + a_1\, p_{r_*}^2/(r\Omega)^2 + a_2\, \ddot{r}/r\,\Omega^2}$$

a_1 & a_2 are determined by requiring:

- The maximum of the (Zerilli-normalized) EOB metric waveform is equal to the maximum of the NR waveform
- That this maximum occurs at the EOB “light-ring” [i.e., maximum of EOB orbital frequency].
- Using **two** NR data: maximum $\boxed{\varphi(\nu) \simeq 0.3215\nu(1 - 0.131(1 - 4\nu))}$
- NQC correction is added consistently in RR. **Iteration until a_1 & a_2 stabilize**

Remaining EOB 2.0 flexibility:

$$A(u; a_5, a_6, \nu) \equiv P_5^1\left[A^{3\mathrm{PN}}(u) + \nu a_5 u^5 + \nu a_6 u^6\right]$$

Use Caltech-Cornell [inspiral-plunge] data to constrain ***(a_5,a_6)***

A wide region of correlated values ***(a_5,a_6)*** exists where the phase difference can be reduced at the level of the numerical error (<0.02 radians) during the inspiral

EOB *metric* gravitational waveform: merger and ringdown

EOB approximate representation of the merger (DRT1972 inspired) :

- sudden change of description around the "EOB light-ring" $t=t_m$ (maximum of orbital frequency)
- "match" the insplunge waveform to a superposition of QNMs of the final Kerr black hole
- matching on a 5-teeth comb (*found efficient in the test-mass limit, DN07a*)
- comb of width around 7M centered on the "EOB light-ring"
- use 5 positive frequency QNMs (found to be near-optimal in the test-mass limit)
- Final BH mass and angular momentum are computed from a fit to NR ringdown (*5 eqs for 5 unknowns*)

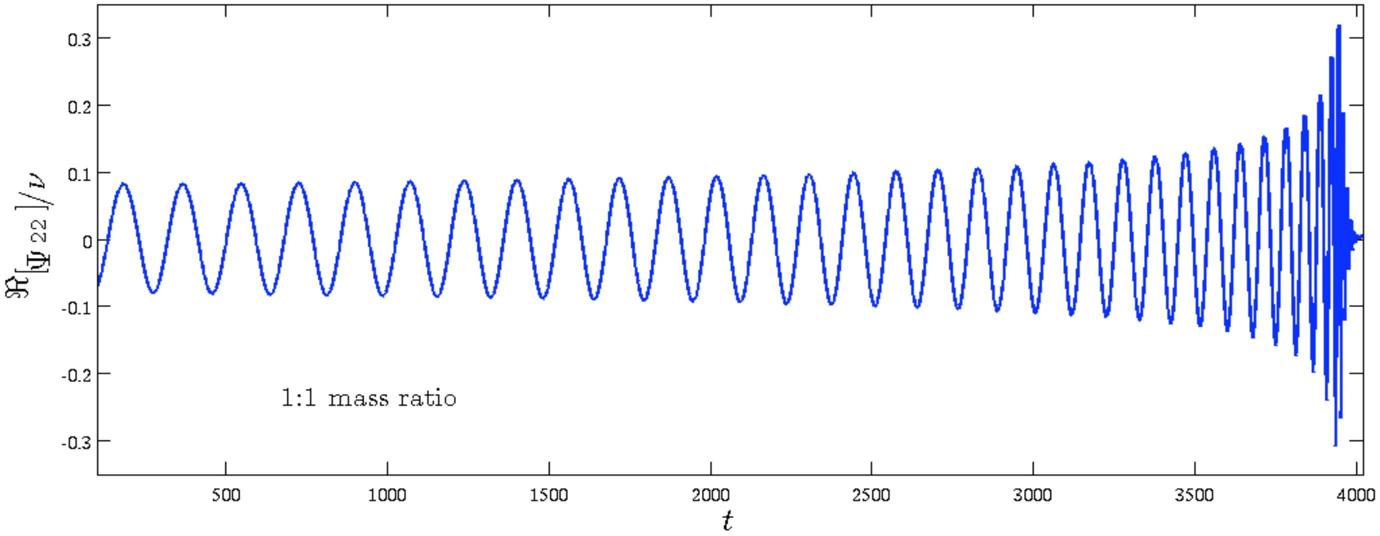
$$\Psi_{22}^{\rm ringdown}(t) = \sum_N C_N^+ e^{-\sigma_N^+ t} \,.$$

Total EOB waveform covering inspiral-merger and ringdown

$$\boxed{h_{22}^{\rm EOB}(t) = \theta(t_m - t)\, h_{22}^{\rm insplunge}(t) + \theta(t - t_m)\, h_{22}^{\rm ringdown}(t)}$$

Binary BH coalescence: Numerical Relativity waveform

1:1 (no spin) Caltech-Cornell simulation. Inspiral: $\Delta\phi$<0.02 rad; Ringdown: $\Delta\phi$~0.05 rad
Boyle et al 07, Scheel et al 09



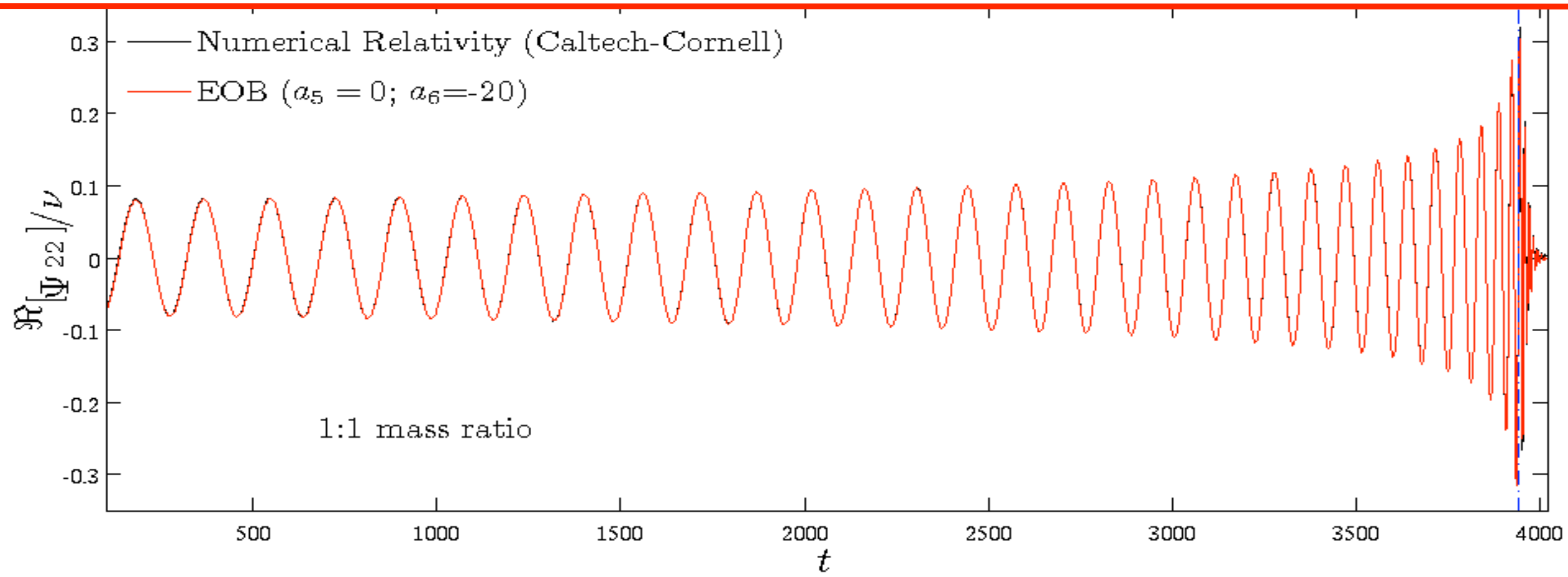
Ringdown

Early inspiral

Late inspiral & Merger

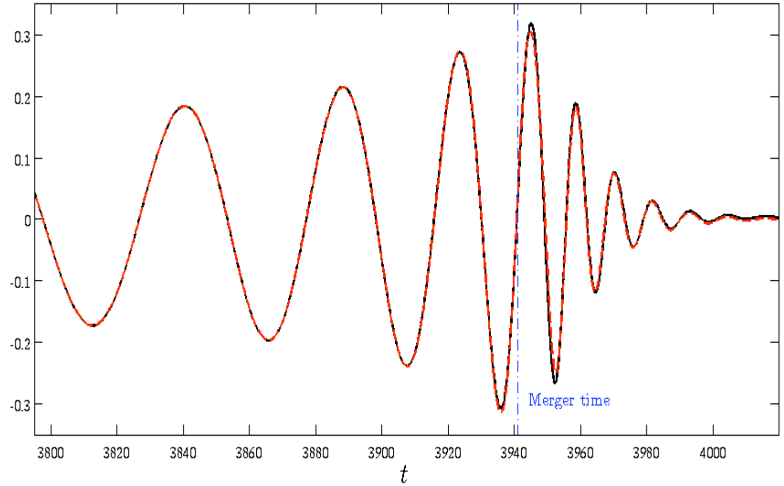
- Late inspiral and merger is non perturbative
- Only describable by NR ?

Comparison Effective-One-Body (EOB) vs NR waveforms



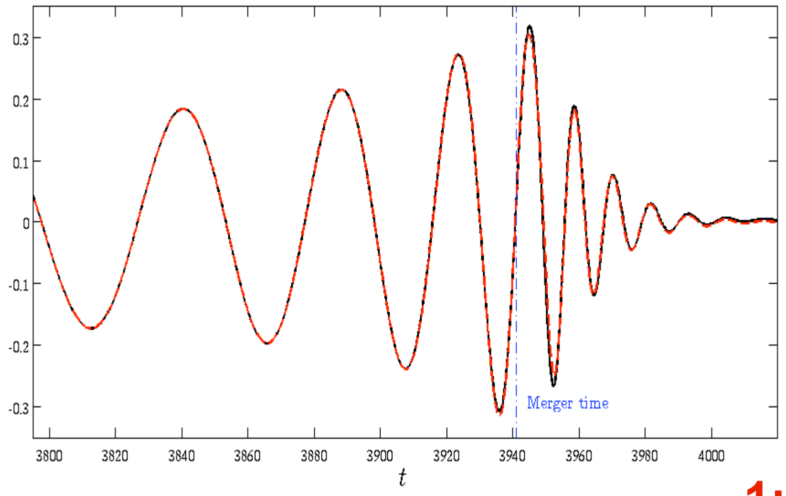
“New” EOB formalism: EOB 2.0$_{NR}$

- Two unknown EOB parameters: 4PN and 5PN effective corrections in 2-body Hamiltonian, ***(a_5,a_6)***
- NR calibration of the maximum GW amplitude
- Need to “tune” only one parameter
- **Banana-like “best region”** in the (a_5,a_6) plane extending from (0,-20) to (-36, 520) (where $\Delta\phi \leq 0.02$)

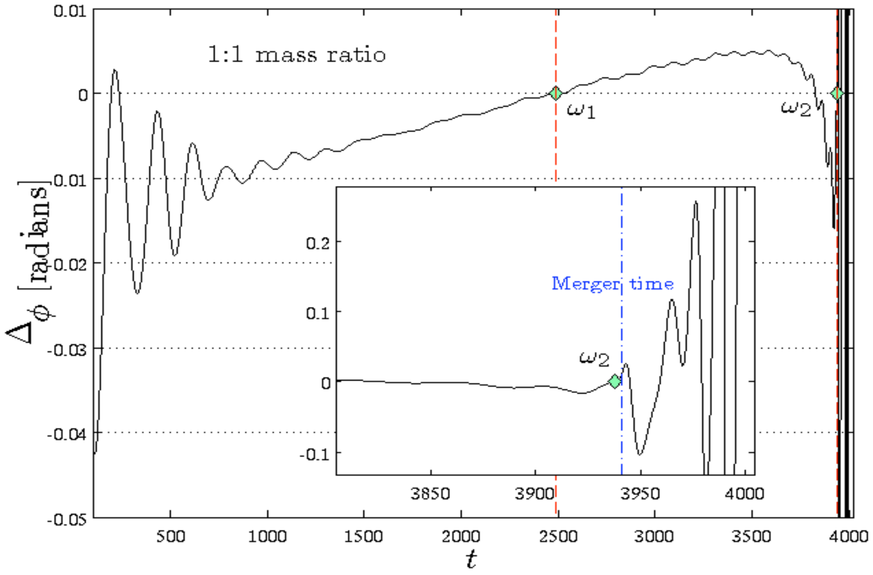


Damour & Nagar, Phys. Rev. D **79**, 081503(R), (2009)
Damour, Iyer & Nagar, Phys. Rev. D **79**, 064004 (2009)

EOB 2.0 & NR comparison: 1:1 & 2:1 mass ratios

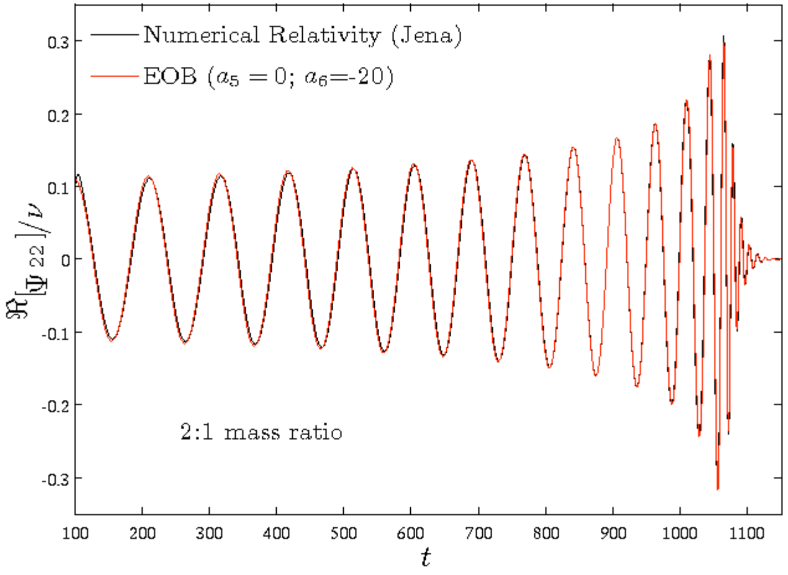


1:1



$a_5 = 0, a_6 = -20$

2:1



D, N, Hannam, Husa, Brügmann 08

EOB 1.5: Buonanno, Pan, Pfeiffer, Scheel, Buchman & Kidder, Phys Rev.D79, 124028 (2009)

- **EOB formalism: EOB 1.5 U NR**

h_{lm} [RWZ] NR 1:1. **EOB resummed waveform** (à la DIN)

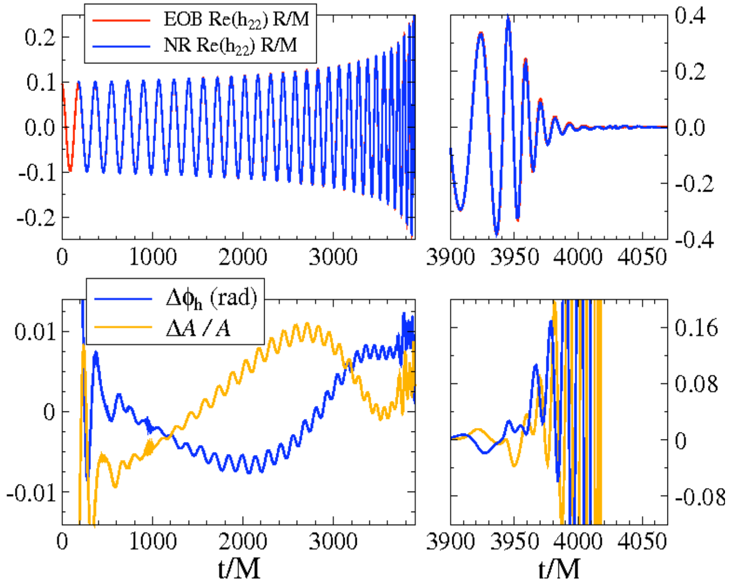
$$\left.\begin{array}{l} a_5 = 25.375 \\ v_{pole}(\nu=1/4) = 0.85 \end{array}\right\}$$ **reference values**

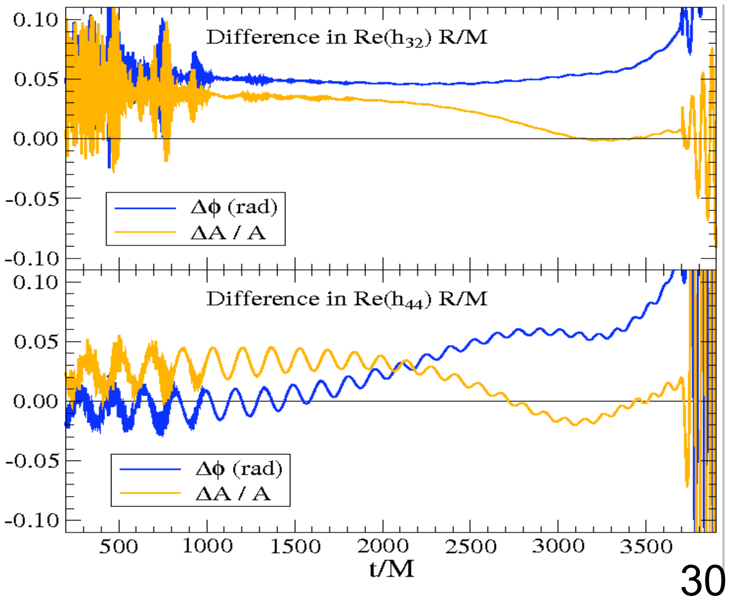
$\Delta t^{22}_{match} = 3.0M$

$a_1 = -2.23$
$a_2 = 31.93$
$a_3 = 3.66$
$a_4 = -10.85$

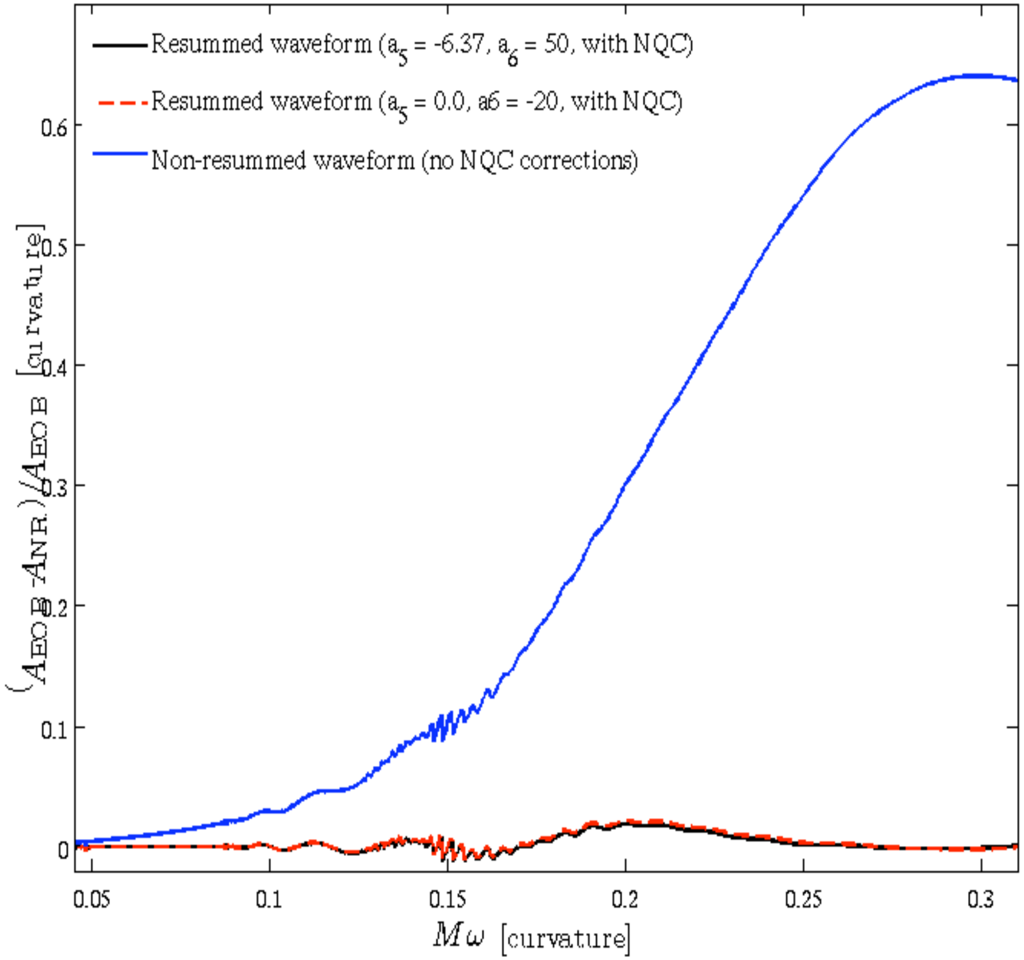
$-0.02 \leq \Delta\phi \leq +0.02 \quad -0.02 \leq DA/A \leq +0.02$ [l=m=2]

- Here, 1:1 mass ratio (**with higher multipoles**)
- **Plus** 2:1 & 3:1 [inspiral only] mass ratios





(Fractional) curvature amplitude difference EOB-NR



- Nonresummed: fractional differences start at the 0.5% level and build up to more than 60%! (just before merger)
- New resummed EOB amplitude+NQC corrections: fractional differences start at the 0.04% level and build up to only 2% (just before merger)
- *Resum+NQC: factor ~30 improvement!*

Shows the effectiveness of resummation techniques, even during (early) inspiral.

Tidal effects and EOB formalism

- tidal effects are important in late inspiral of binary neutron stars

Flanagan, Hinderer 08, Hinderer et al 09, Damour, Nagar 09, Binnington, Poisson 09

⟶ a possible handle on the nuclear equation of state

- tidal extension of EOB formalism : non minimal worldline couplings

$$\Delta S_{\mathrm{nonminimal}} = \sum_A \frac{1}{4}\,\mu_2^A \int ds_A (u^\mu u^\nu R_{\mu\alpha\nu\beta})^2 + \ldots$$

Damour, Esposito-Farèse 96, Goldberger, Rothstein 06, Damour, Nagar 09

⟶ modification of EOB effective metric + … :

$$\begin{aligned} A(r) &= A^0(r) + A^{\mathrm{tidal}}(r) \\ A^{\mathrm{tidal}}(r) &= -\kappa_2\, u^6 (1 + \bar{\alpha}_1\, u + \bar{\alpha}_2\, u^2 + \ldots) + \ldots \end{aligned}$$

- need accurate NR simulation to "calibrate" the higher-order PN contributions that are quite important during late inspiral

  Uryu et al 06, 09, Rezzolla et al 09

Conclusions (1)

- Analytical Relativity : though we are far from having mathematically rigorous results, there exist perturbative calculations that have obtained unambiguous results at a high order of approximation (3 PN ~ 3 loops). They are based on a "cocktail" of approximation methods : post-Minkowskian, post-Newtonian, multipolar expansions, matching of asymptotic expansions, use of effective actions, analytic regularization, dimensional regularization,…

- Numerical relativity : Recent breakthroughs (based on a "cocktail" of ingredients : new formulations, constraint damping, punctures, …) allow one to have an accurate knowledge of nonperturbative aspects of the two-body problem.

- There exists a complementarity between Numerical Relativity and Analytical Relativity, especially when using the particular resummation of perturbative results defined by the Effective One Body formalism. The NR- tuned EOB formalism is likely to be essential for computing the many thousands of accurate GW templates needed for LIGO/Virgo/GEO.

Conclusions (2)

- There is a synergy between AR and NR, and many opportunities for useful interactions : arbitrary mass ratios, spins, extreme mass ratio limit, tidal interactions,…

- The two-body problem in General Relativity is more lively than ever. This illustrates Poincaré's sentence :

“*Il n'y a pas de problèmes résolus,*
il y a seulement des problèmes plus ou moins résolus”.