# Some analytical calculations of caustics

Let's consider the elliptical isothermal potential for small values of the elliptical parameter $\eta$

$$\phi=\sqrt{(1-\eta)x^2+(1+\eta)y^2}\simeq r\left(1-\frac{\eta}{2}\cos(2\theta)\right)$$

Then the lens equation $\vec{r}_S=\vec{r}-\vec{\nabla}\phi$ Implies that:

$$\begin{cases} x_S=x-\dfrac{\partial\phi}{\partial x}\simeq r\cos(\theta)-\cos(\theta)+\dfrac{\eta}{4}(3\cos(\theta)-\cos(3\theta)) \\ y_S=y-\dfrac{\partial\phi}{\partial y}\simeq r\sin(\theta)-\sin(\theta)+\dfrac{\eta}{4}(-3\sin(\theta)-\sin(3\theta)) \end{cases}$$

The Jacobian matrix J relates the volume elements in different coordinates:

$$d x_s d y_s = |J| d r d \theta \qquad J = \begin{pmatrix} \frac{\partial x_s}{\partial r} & \frac{\partial y_s}{\partial r} \\ \frac{\partial x_s}{\partial \theta} & \frac{\partial y_s}{\partial \theta} \end{pmatrix}$$

The the amplification A is: $\frac{d x d y}{d x_s d y_s} = \frac{r\, dr\, d \theta}{d x_s d y_s} = \frac{r}{|J|}$

As a consequence an infinite amplification corresponds to $|J| = 0$

To first order in $\eta$, $|J| \simeq r - 1 - \frac{3}{2} \eta \cos(2\theta)$, $Thus$ $|J| = 0 \Leftrightarrow r = 1 + \frac{3}{2} \eta \cos(2\theta)$

Thus for an infinite amplification the location of the images (the critical lines) to first order is an ellipse.

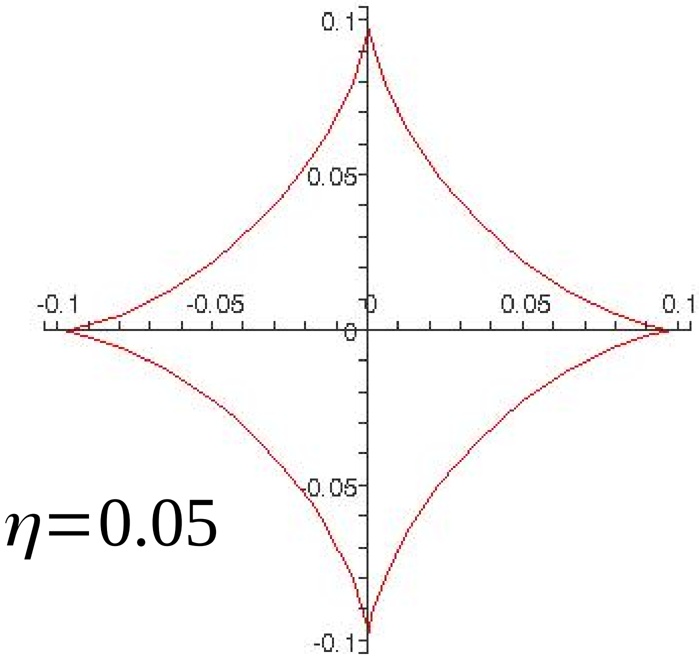
# The caustic lines

Putting back the critical line equation and transforming to the lens plane
Using the lens equation:

$$\begin{cases} x_s = x - \dfrac{\partial \phi}{\partial x} \simeq r\cos(\theta) - \cos(\theta) + \dfrac{\eta}{4}\left(3\cos(\theta) - \cos(3\theta)\right) \\ y_s = y - \dfrac{\partial \phi}{\partial y} \simeq r\sin(\theta) - \sin(\theta) + \dfrac{\eta}{4}\left(-3\sin(\theta) - \sin(3\theta)\right) \\ r = 1 + \dfrac{3}{2}\eta\cos(2\theta) \end{cases}$$

Leading to:

$$\begin{cases} x_s \simeq \dfrac{\eta}{2}\left(3\cos(\theta) + \cos(3\theta)\right) \\ y_s \simeq \dfrac{\eta}{2}\left(-3\sin(\theta) + \sin(3\theta)\right) \end{cases}$$




$\eta = 0.05$