Exploring the weak-lensing reconstruction of caustics

Let's consider the elliptical isothermal potential for small values of the elliptical parameter η

$$\phi = R_E \sqrt{(1-\eta(x,y))\,x^2 + (1+\eta(x,y))\,y^2} \simeq R_E\, r\left(1 - \frac{\eta(r,\theta)}{2}\cos(2\theta)\right)$$

We will work at a large distance, $r \gg R_E$ in a small area around the image in this area we will make the approximation that: $\eta(r,\theta) \simeq \eta_0 = C^{ste}$

The local distortion of the image of galaxies is directly related to the Jacobian matrix

$$J = \begin{pmatrix} 1 - \dfrac{\partial^2 \phi}{\partial x^2} & \dfrac{\partial^2 \phi}{\partial x \partial y} \\ \dfrac{\partial^2 \phi}{\partial x \partial y} & 1 - \dfrac{\partial^2 \phi}{\partial y^2} \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 - g_1 & -g_2 \\ -g_2 & 1 + g_1 \end{pmatrix}$$

The second order moments of the local galaxies $Q_{ij} = \int \Sigma(\vec{r}) x_i x_j d^2 x$

And the associated Stokes parameters $\alpha = \dfrac{Q_{22} - Q_{11}}{Q_{11} + Q_{22}}$ $\beta = \dfrac{2 Q_{21}}{Q_{11} + Q_{22}}$

The weak lensing distortion represented by the Jacobian Matrix J transformation of the Stokes parameters reads:

$$\alpha_S = \alpha + g_1 \qquad \beta_S = \beta + g_2$$

Re-writing explicitly the Jacobian matrix using the reduced shear

$$J = \begin{pmatrix} 1-\phi_{11} & -\phi_{12} \\ -\phi_{12} & 1-\phi_{22} \end{pmatrix} = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix} = (1-\kappa)\begin{pmatrix} 1-g_1 & -g_2 \\ -g_2 & 1+g_1 \end{pmatrix}$$

$$\kappa = \frac{1}{2}(\phi_{11}+\phi_{22}) \qquad \gamma_1 = \frac{1}{2}(\phi_{11}-\phi_{22}) \qquad \gamma_2 = \phi_{12}$$

$$g_i = \frac{\gamma_i}{(1-\kappa)}$$ (reduced shear)

For our local potential $\phi=R_E\sqrt{(1-\eta_0)\,x^2+(1+\eta_0)\,y^2}$ and for $\eta_0 \ll 1,\quad \frac{R_E}{r}\ll 1$

Using the equivalent substitution: $\eta_0=\epsilon\eta_0,\quad r=\frac{r}{\epsilon},\epsilon\ll 1$

And expanding to second order in ϵ

$$\kappa\simeq\frac{R_E}{4r}\left(2+3\,\eta\cos\left(2\,\theta\right)\right)$$

$$\gamma_1\simeq-\ \frac{R_E}{4r}\cos\left(2\,\theta\right)\left(2+3\,\eta\cos\left(2\,\theta\right)\right)$$

$$\gamma_2\simeq-\ \frac{R_E}{4r}\sin\left(2\,\theta\right)\left(2+3\,\eta\cos\left(2\,\theta\right)\right)$$

$$g_1\simeq-\ \frac{R_E}{4r}\cos\left(2\,\theta\right)\left(2+3\,\eta\cos\left(2\,\theta\right)+\frac{R_E}{r}\right)$$

$$g_2\simeq-\ \frac{R_E}{4r}\sin\left(2\,\theta\right)\left(2+3\,\eta\cos\left(2\,\theta\right)+\frac{R_E}{r}\right)$$

Adopting a complex representation for the shear vector: $G = g_1 + i g_2 = g_S \exp(i\theta_S)$

$$g_S = \sqrt{g_1^2 + g_2^2} \simeq \frac{R_E}{4r}\left(2 + 3\cos(2\theta)\eta_0 + \frac{R_E}{r}\right) \quad ; \qquad \tan(\theta_S) = \tan(2\theta)$$

All the information is in g_S leading to the following decomposition of g_S

The lowest order term $q_0 = \frac{R_E}{2r}$ contains the mass ratio, $q_0^2 \propto \frac{M_E}{r^2}$

The next term contains the ellipticity parameter $q_1 = \frac{R_E}{4r}\left(3\cos(2\theta)\eta_0 + \frac{R_E}{r}\right)$

The effective ellipticity term can be separated by analyzing the angular dependence as a function of the $\cos(2\theta)$ term. Note that the amplitude of this term Is much smaller that the former term. The effective ellipticity of the potential is smaller than that of the density, and at large distances (the weak lensing regime) the ellipticity tends to decrease quickly.

The problem with the practical estimation of the potential parameters

The lowest order term $q_0 = \frac{R_E}{2r}$ is typically about a few percent

$$q_1 = \frac{R_E}{4r}\left(3\cos(2\theta)\,\eta_0 + \frac{R_E}{r}\right) \longrightarrow$$

The amplitude of the term associated with the ellipticity is about $q_0\,\eta_0$

At larger distances the potential become almost circular and η_0 is of the order of a percent. As consequence $q_0\,\eta_0$ is only of the order a fraction of percent

As a consequence any small bias in the estimation of the moment will dramatically affect the estimation of the potential ellipticity

Small biases in the PSF model may compromise the estimations

In astronomical images: the images of galaxies are convolved with the PSF

It is not difficult to show that the second order moments of the convolved object $Q_{i,j}^{C}$ are related to the second order moments of the un-convolved image $Q_{i,j}$ and of the PSF $Q_{i,j}^{PSF}$

$$Q_{i,j}^{C} = Q_{i,j} + Q_{i,j}^{PSF}$$

As a consequence any small bias in the reconstruction of the PSF
Moments, event of the order of a fraction of a percent can definitely
Compromise the evaluation of the potential ellipticity. Knowing the PSF
With exceedingly good accuracy is absolutely essential. Any failure in the
Reconstruction of the PSF and especially of its variability as a function of the
Position in the image may dramatically affect the estimation of the potential

Considering an image: $I(\vec{x})$ and a centered and normalized PSF model: $\psi(\vec{x})$

The image convolved with the PSF $I^C(\vec{x})$ reads, $I^C(\vec{x}) = \int I(\vec{x}-\vec{u})\,\psi(\vec{u})\,d^2\vec{u}$

With: PSF normalization $\int \psi(\vec{u})\,d^2\vec{u} = 1$, centering, $\int \psi(\vec{u})\,u_i\,d^2\vec{u} = 0$

The second order moments of $I_x(\vec{x})$ are, $Q_{i,j} = \frac{P_{i,j}}{S} = \frac{\int I^C(\vec{x})\,x_i x_j\,d^2\vec{x}}{\int I^C(\vec{x})\,d^2\vec{x}}$

With $\vec{z} = \vec{x} - \vec{u}$

$$S = \int I^C(\vec{x})\,d^2\vec{x} = \int I(\vec{x}-\vec{u})\,\psi(\vec{u})\,d^2\vec{u}\,d^2\vec{x} = \int I(\vec{z})\,\psi(\vec{u})\,d^2\vec{u}\,d^2\vec{x} = \int I(\vec{x})\,d^2\vec{x}$$

Similarly,

$$P_{i,j}=\int I^C(\vec{x})x_i x_j d^2\vec{x}=\int I(\vec{x}-\vec{u})\,\psi(\vec{u})x_i x_j d^2\vec{u}\,d^2\vec{x}=\int I(\vec{z})\,\psi(\vec{u})(z_i+u_i)(z_j+u_j)d^2\vec{u}\,d^2\vec{z}$$

$$P_{i,j}=\int I(\vec{z})\,\psi(\vec{u})z_i z_j d^2\vec{u}\,d^2\vec{z}=\int I(\vec{z})z_i z_j d^2\vec{z}$$

$$+\int I(\vec{z})\,\psi(\vec{u})z_i u_j d^2\vec{u}\,d^2\vec{z}=0$$

$$+\int I(\vec{z})\,\psi(\vec{u})z_j u_i d^2\vec{u}\,d^2\vec{z}=0$$

$$+\int I(\vec{z})\,\psi(\vec{u})u_i u_j d^2\vec{u}\,d^2\vec{z}=\int I(\vec{z})d^2\vec{z}\int\psi(\vec{u})u_i u_j d^2\vec{u}=S\,Q^{PSF}_{i,j}$$

Leading to: $$Q^C_{i,j}=\frac{P^C_{i,j}}{S}=Q_{i,j}+Q^{PSF}_{i,j}$$