FORMULATIONS OF EINSTEIN EQUATIONS

Isabel Cordero-Carrión

Max-Plank Institute of Astrophysics, Garching

Institut d'Astrophysique de Paris

Paris, November 2012

Alexander von Humboldt Stiftung/Foundation

Max Planck Institute for Astrophysics

Outline

· Why general relativity?

· Different formalisms.

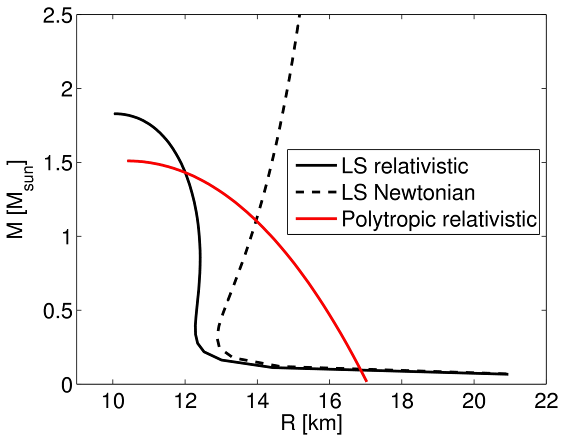
· 3+1 formalisms:

- Free formulations: BSSN.
- Constrained formulations: CFC, xCFC and FCF.

· PIRK methods. Evolution of wave-like equations (hyperbolic sectors) in FCF and BSSN.

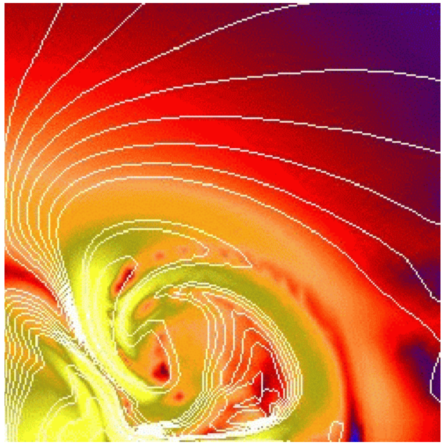
· Conclusions and future work.

Why General Relativity? Some astrophysical scenarios involving compact objects do need a relativistic treatment of gravity

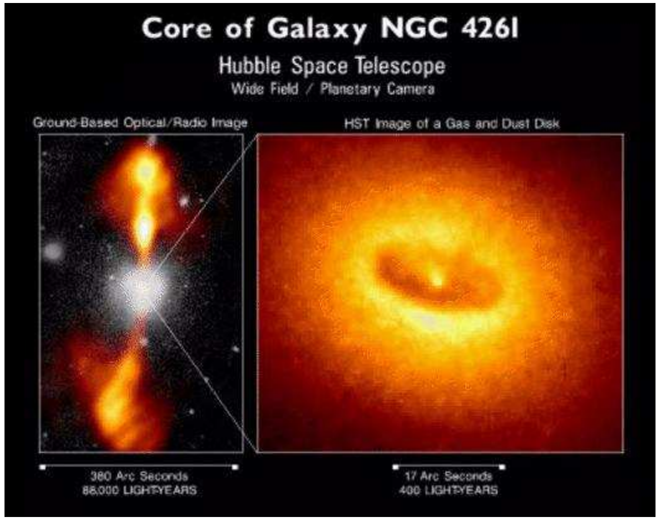


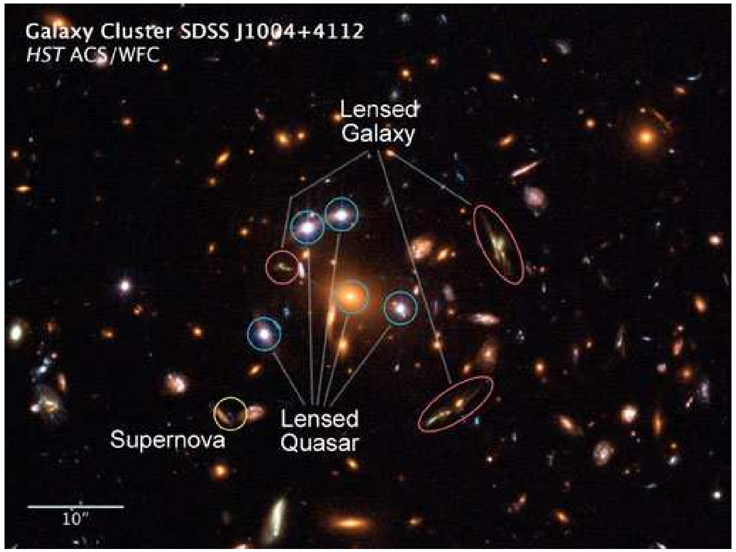
Maximum stellar mass in equilibrium configurations

Magnetorotational Supernova Core Collapse (*Cerdá-Durán et al. Astron. Astrophys., 474, 2007*)



Strong gravity and black holes, accretion disks, jets…





Gravitational lensing

Formalisms of Einstein equations: the appropriate tool

$$G(\boldsymbol{g}) := Ric(\boldsymbol{g}) - \frac{R(\boldsymbol{g})}{2}\boldsymbol{g} = \kappa \boldsymbol{T}$$

Very nice compact formula but… non-linear partial differential equations, coupled with evolution of matter fields (non-vacuum), neutrinos...

- 2+2 formalism: null cones, analysis of radiation at spatial and null infinity.

- 3+1 formalism (Lichnerowitz 1944, Choquet-Bruhat 1952): spacetime is foliated by spacelike hypersurfaces, and an evolution through different hypersurfaces is performed.

· Most common used in numerical relativity.

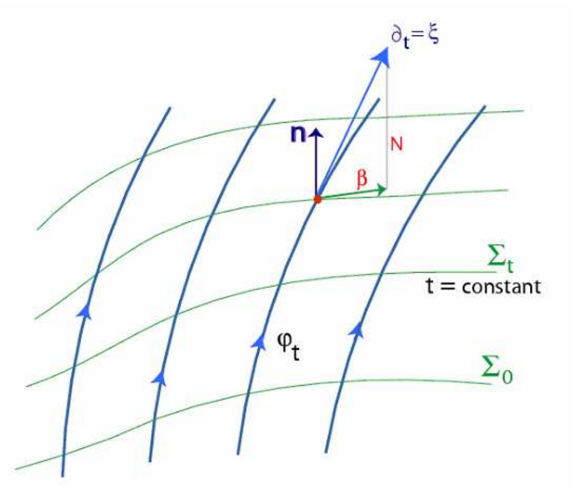
· **Gauge freedom**: some variables can be freely chosen in order to consider the more desired/convenient foliation of spacetime:

* Physical motivations: e.g., maximal slicing in order to avoid singularities…

* Mathematical motivations: e.g., well-posedness of the resulting PDE system…

* Numerical motivations: e.g., stability of long-term numerical simulations…

Decomposition of Einstein equations



Definition of lapse function and shift vector:

$$\xi = N\boldsymbol{n} + \boldsymbol{\beta}.$$

Definition of 3-metric onto spatial hypersurface:

$$\gamma = \gamma_{ij} dx^i dx^j$$

Metric line element:

$$g_{\alpha\beta} dx^\alpha dx^\beta = -N^2 dt^2 + \gamma_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right)$$

Extrinsic curvature:

$$\boldsymbol{K} := -\frac{1}{2}\mathcal{L}_n \gamma$$

* ij components: evolution equations (12 = 6 + 6): 3-metric and extrinsic curvature:

$\partial_t \gamma_{ij} = -2\alpha K_{ij}$ + advection term

$\partial_t K_{ij} =$ advection term + curvature sources + matter sources

* 00 and 0i components: constraint equations (4): Hamiltonian and momentum constraints.

* Non vacuum spacetimes: continuity equation, Bianchi identities, Maxwell equations, equation of state…

Free evolution versus constrained evolution schemes

If the constraints equations are fulfilled initially, then they are fulfilled ANALYTICALLY during the evolution…

* Free evolution schemes: evolve only the evolution equations and MONITOR numerical error in the constraints…

- BSSN: Baumgarte and Shapiro, 1998; Shibata and Nakamura, 1995.

$$\left(\frac{\partial}{\partial t} - \mathscr{L}_{\boldsymbol{\beta}}\right)\Psi = \frac{\Psi}{6}\left(\tilde{D}_i\beta^i - NK\right)$$

$$\left(\frac{\partial}{\partial t} - \mathscr{L}_{\boldsymbol{\beta}}\right)K = -\Psi^{-4}\left(\tilde{D}_i\tilde{D}^iN + 2\tilde{D}_i\ln\Psi\,\tilde{D}^iN\right) + N\left[4\pi(E+S) + \tilde{A}_{ij}\tilde{A}^{ij} + \frac{K^2}{3}\right]$$

$$\left(\frac{\partial}{\partial t} - \mathscr{L}_{\boldsymbol{\beta}}\right)\tilde{\gamma}_{ij} = -2N\tilde{A}_{ij} - \frac{2}{3}\tilde{D}_k\beta^k\,\tilde{\gamma}_{ij}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \mathscr{L}_{\boldsymbol{\beta}}\right)\tilde{A}_{ij} = & -\frac{2}{3}\tilde{D}_k\beta^k\,\tilde{A}_{ij} + N\left[K\tilde{A}_{ij} - 2\tilde{\gamma}^{kl}\tilde{A}_{ik}\tilde{A}_{jl} - 8\pi\left(\Psi^{-4}S_{ij} - \frac{1}{3}S\tilde{\gamma}_{ij}\right)\right]\\ & +\Psi^{-4}\Bigg\{-\tilde{D}_i\tilde{D}_jN + 2\tilde{D}_i\ln\Psi\,\tilde{D}_jN + 2\tilde{D}_j\ln\Psi\,\tilde{D}_iN\\ & +\frac{1}{3}\left(\tilde{D}_k\tilde{D}^kN - 4\tilde{D}_k\ln\Psi\,\tilde{D}^kN\right)\tilde{\gamma}_{ij}\\ & +N\Bigg[\frac{1}{2}\left(-\tilde{\gamma}^{kl}\bar{D}_k\bar{D}_l\tilde{\gamma}_{ij} + \tilde{\gamma}_{ik}\bar{D}_j\tilde{\Gamma}^k + \tilde{\gamma}_{jk}\bar{D}_i\tilde{\Gamma}^k\right) + \mathscr{Q}_{ij}(\tilde{\boldsymbol{\gamma}},\bar{\boldsymbol{D}}\tilde{\boldsymbol{\gamma}})\\ & -\frac{1}{3}\left(\bar{D}_k\tilde{\Gamma}^k + \mathscr{Q}(\tilde{\boldsymbol{\gamma}},\bar{\boldsymbol{D}}\tilde{\boldsymbol{\gamma}})\right)\tilde{\gamma}_{ij} - 2\tilde{D}_i\tilde{D}_j\ln\Psi + 4\tilde{D}_i\ln\Psi\,\tilde{D}_j\ln\Psi\\ & +\frac{2}{3}\left(\tilde{D}_k\tilde{D}^k\ln\Psi - 2\tilde{D}_k\ln\Psi\,\tilde{D}^k\ln\Psi\right)\tilde{\gamma}_{ij}\Bigg]\Bigg\}.\end{aligned}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \mathscr{L}_{\boldsymbol{\beta}}\right)\tilde{\Gamma}^i = & \frac{2}{3}\bar{D}_k\beta^k\,\tilde{\Gamma}^i + \tilde{\gamma}^{jk}\bar{D}_j\bar{D}_k\beta^i + \frac{1}{3}\tilde{\gamma}^{ij}\bar{D}_j\bar{D}_k\beta^k - 2\tilde{A}^{ij}\bar{D}_jN\\ & -2N\left[8\pi\Psi^4p^i - \tilde{A}^{jk}\Delta^i{}_{jk} - 6\tilde{A}^{ij}\bar{D}_j\ln\Psi + \frac{2}{3}\tilde{\gamma}^{ij}\bar{D}_jK\right],\end{aligned}$$

$$\tilde{\Gamma}^i + \bar{D}_j\tilde{\gamma}^{ij} = 0$$

Free evolution versus constrained evolution schemes

If the constraints equations are fulfilled initially, then they are fulfilled ANALYTICALLY during the evolution…

* Free evolution schemes: evolve only the evolution equations and MONITOR numerical error in the constraints…

- BSSN: Baumgarte and Shapiro, 1998; Shibata and Nakamura, 1995.

Gauge: Evolution equations for lapse and shift…

Do not forget all the constraints: Hamiltonian, momentum, determinant of the conformal metric (fix the value of the conformal factor), tracessless part of the conformal extrinsic curvature (valid decomposition of extrinsic curvature), definition of extra variables:

$$\tilde{D}_i \tilde{D}^i \Psi - \frac{1}{8} \tilde{R} \Psi + \left(\frac{1}{8} \tilde{A}_{ij} \tilde{A}^{ij} - \frac{1}{12} K^2 + 2\pi E \right) \Psi^5 = 0$$

$$\tilde{D}^j \tilde{A}_{ij} + 6 \tilde{A}_{ij} \tilde{D}^j \ln \Psi - \frac{2}{3} \tilde{D}_i K = 8\pi p_i$$

$$\det(\tilde{\gamma}_{ij}) = f$$

$$\tilde{\gamma}^{ij} \tilde{A}_{ij} = 0$$

$$\tilde{\Gamma}^i + \tilde{D}_j \tilde{\gamma}^{ij} = 0.$$

Free evolution versus constrained evolution schemes

If the constraints equations are fulfilled initially, then they are fulfilled ANALYTICALLY during the evolution…

* Free evolution schemes: evolve only the evolution equations and MONITOR numerical error in the constraints…

- BSSN: Baumgarte and Shapiro, 1998; Shibata and Nakamura, 1995.

· Most common used in numerical relativity.

· Development of ‘damping terms’ to control the growth of the violation of the constraints: Possible (numerical) violation of the constraints is included in the evolution equations (equivalent analytical system) inducing a fast (exponentially) decay behavior. Addition of new parameters.

· Essential for binary black hole simulations using the ‘moving puncture’ for the BH.

Free evolution versus constrained evolution schemes

If the constraints equations are fulfilled initially, then they are fulfilled ANALYTICALLY during the evolution…

* Free evolution schemes: evolve only the evolution equations and MONITOR numerical error in the constraints…

- BSSN: Baumgarte and Shapiro, 1998; Shibata and Nakamura, 1995.

· Most common used in numerical relativity.

· Development of 'damping terms' to control the growth of the violation of the constraints: Possible (numerical) violation of the constraints is included in the evolution equations (equivalent analytical system) inducing a fast (exponentially) decay behavior. Addition of new parameters.

· Essential for binary black hole simulations using the 'moving puncture' for the BH.

* Constrained evolution schemes: evolve the evolution equations and SOLVE the constraints equations on each slice…

- CFC (Conformally Flat Condition): Isenberg, 1979/2008; Wilson and Mathews, 1989.
- FCF (Fully Constrained Formulation): Bonazzola et al., 2004.

CFC (Conformally Flat Condition)

· Approximation to Einstein equations: spatial metric is assumed to be conformally flat. The difference of the conformal physical metric and the flat one is neglected; the gravitational radiation is encoded in these terms.

· Exact approach in the case of spherical symmetry (C.-C. et al., 2011, constructive proof).

· Very accurate for axisymmetric rotating neutron stars.

· Correct at the 1-PN order in the post-Newtonian expansion of general relativity.

· Set of elliptic equations for the metric variables, $f = (\psi,\ N,\ \beta^i)$, including the constraint equations.

· Used in a lot of applications:

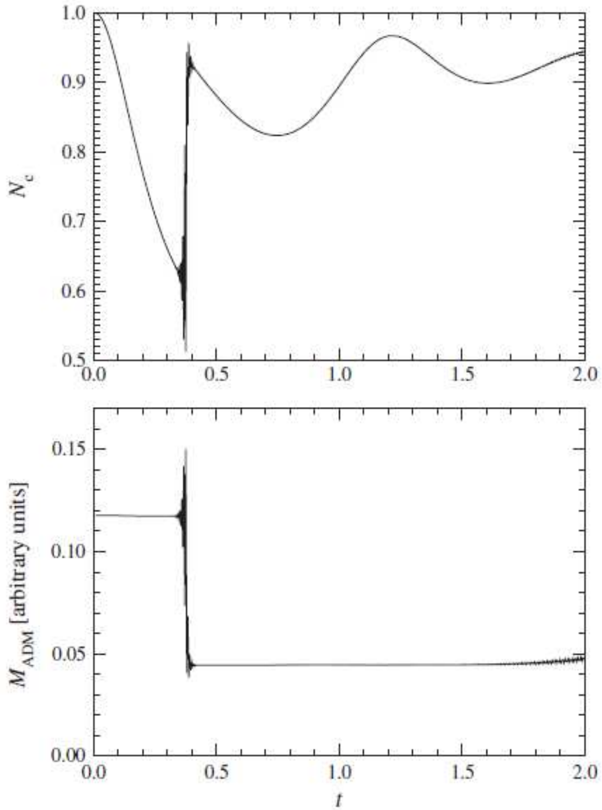
- H. Dimmelmeier & CoCoNuT code: Collapse of rotating cores of massive stars and gravitational waves catalog.

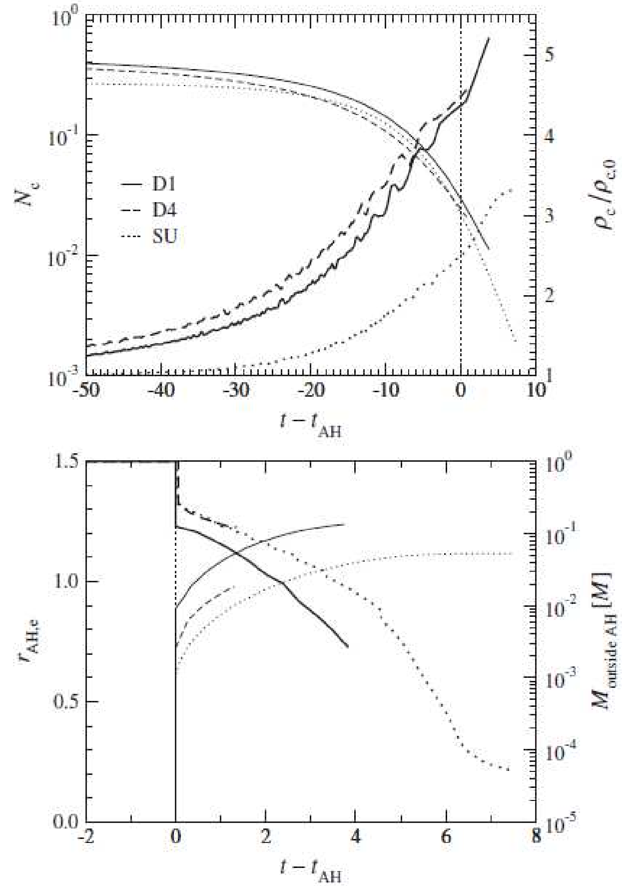
- P. Cerdá-Durán: Equilibrium model of rotating neutron stars.

- A. Bauswein: Evolution of binary compact objects, NS-NS/BH. Necessity of recent new approach of CFC…

xCFC (Extended Conformally Flat Condition)

· Non-local uniqueness pathology in the CFC scheme: extreme curvature or very high density regimes.





· Derivation of the xCFC (C.-C. et al., 2009): reformulation of the CFC scheme introducing an extra vector elliptic equation to overcome the problem.

· xCFC implemented in the X-ECHO code (Bucciantini and Del Zanna, 2011).

FCF (Fully Constrained Formulation)

· Fixed gauge: maximal slicing (K=0) and Dirac generalized gauge (zero divergence of conformal metric with respect to the flat one).

· Maximum number of elliptic equations. The constraints equations result in elliptic ones which are SOLVED.

· The structure of the equations allows for approximations, e.g. based on post-Newtonian estimations.

· Natural extension of the CFC / xCFC approximations: similar elliptic system plus hyperbolic system for gravitational radiation:

$$\Delta f = S_{\rm CFC}(f) + S_f(f,h)$$

$$+$$

$$\frac{\partial^2 h^{ij}}{\partial t^2} - \frac{N^2}{\psi^4}\tilde{\gamma}^{kl}\mathcal{D}_k\mathcal{D}_l h^{ij} - 2\mathcal{L}_\beta \frac{\partial h^{ij}}{\partial t} + \mathcal{L}_\beta\mathcal{L}_\beta h^{ij} = S_h^{ij}$$

Improvements on CFC / xCFC scheme can be automatically introduced.

· Elliptic equations are more stable but difficult to solve and parallelize.

* LORENE: Computational tool able to solve elliptic equations in a very accurate way. Parallelization in progress.

From CFC to FCF: hyperbolic system

$$\frac{\partial^2 h^{ij}}{\partial t^2} - \frac{N^2}{\psi^4}\tilde{\gamma}^{kl}\mathcal{D}_k\mathcal{D}_l h^{ij} - 2\mathcal{L}_\beta \frac{\partial h^{ij}}{\partial t} + \mathcal{L}_\beta\mathcal{L}_\beta h^{ij} = S_h^{ij}$$

· Results from CFC / xCFC are robust and can be applied to the elliptic equations automatically. Experience solving them numerically.

· New ingredient: gravitational radiation encoded in the second order (in time and space) evolution equations:

* Characteristic structure analyzed when the system is written as a first order one in time and space (C.-C. et al., 2008): hyperbolic system; eigenvalues related with speed of light.

* Application to boundary conditions on trapping horizons (Gourgoulhon, Jaramillo, C.-C. and Ibáñez, 2008): radiation towards the BH.

* Numerical stable evolution for neutron stars in 2D and spherical coordinates (C.-C. et al. 2012)… more comments in next slices.

· Coupling of elliptic and hyperbolic equations: numerical evolution of the hyperbolic equations consistent with gauge conditions… work in progress.

FCF: hyperbolic system and gravitational radiation

$$\frac{\partial^2 h^{ij}}{\partial t^2} - \frac{N^2}{\psi^4}\tilde{\gamma}^{kl}\mathcal{D}_k\mathcal{D}_l h^{ij} - 2\mathcal{L}_\beta \frac{\partial h^{ij}}{\partial t} + \mathcal{L}_\beta\mathcal{L}_\beta h^{ij} = S_h^{ij}$$

Numerical evolution of the hyperbolic sector of FCF using the CoCoNuT code (C.-C. et al. 2012):

· 4th order spatial derivatives.

· Spherical coordinates

· Evolution of the hyperbolic sector: 2nd / 4th order PIRK methods.

· Elliptic equations solved in the xCFC approximation (gravitational radiation neglected in the elliptic sector): LORENE library (spectral methods).

· Evolution of GRHD (non vacuum spacetimes): HRSC methods.

· Kreiss-Oliger dissipative term to avoid high frequency noise development.

· Inner boundaries: ghost cells imposing symmetries of the system.

· Outer boundary: Sommerfeld condition.

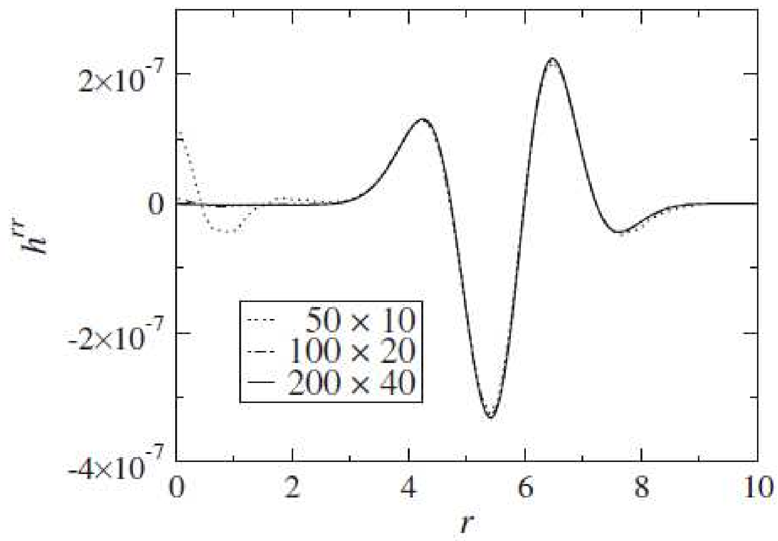
· Gravitational waves extracted using the Weyl scalars.

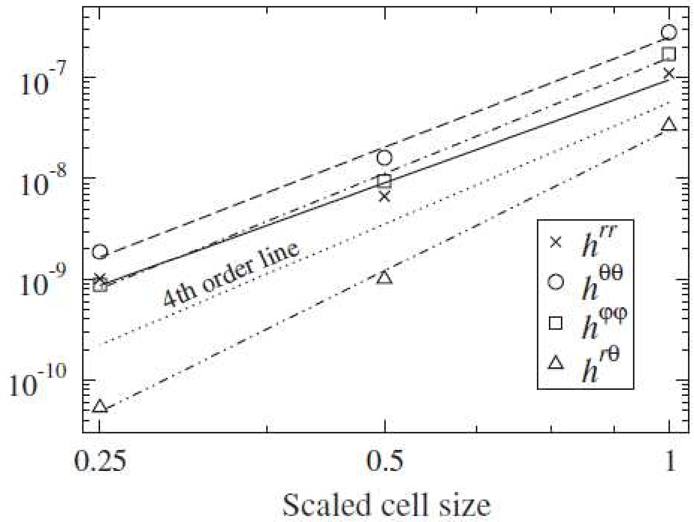
FCF: hyperbolic system and gravitational radiation

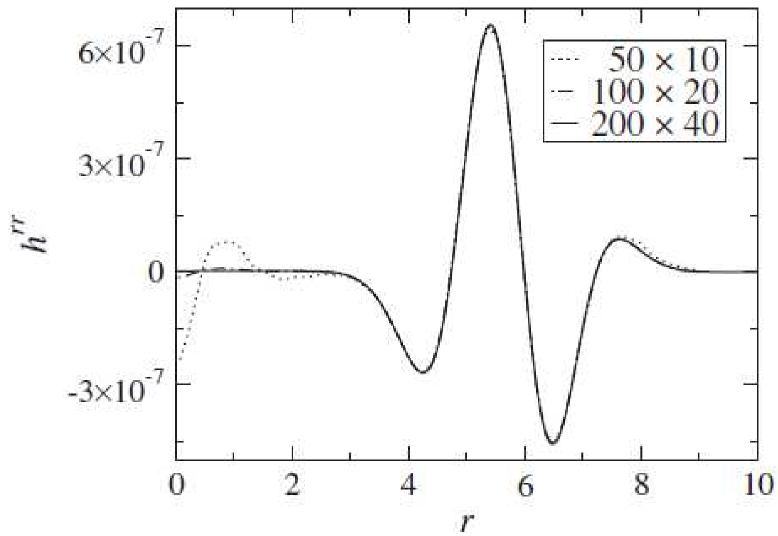
- Teukolsky waves:

Stable numerical evolution in vacuum spacetime (hrr, t=6, equator and pole).

Expected order of convergence.



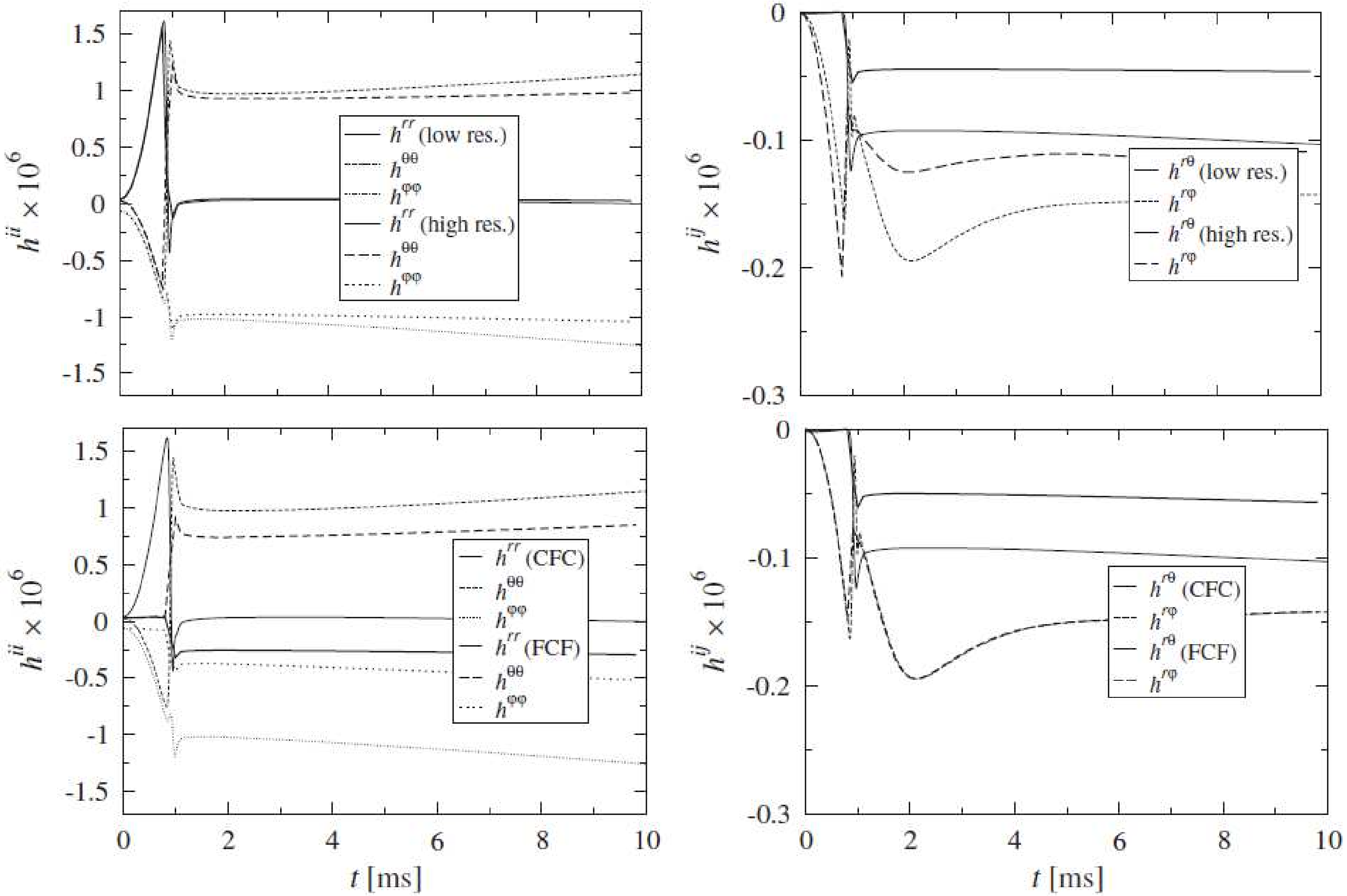




FCF: hyperbolic system and gravitational radiation

- Equilibrium rotating neutron stars:

Influence of accuracy due to spectral resolution and due to xCFC approximation in the elliptic equations (r=19.44 r*, θ=π/2).



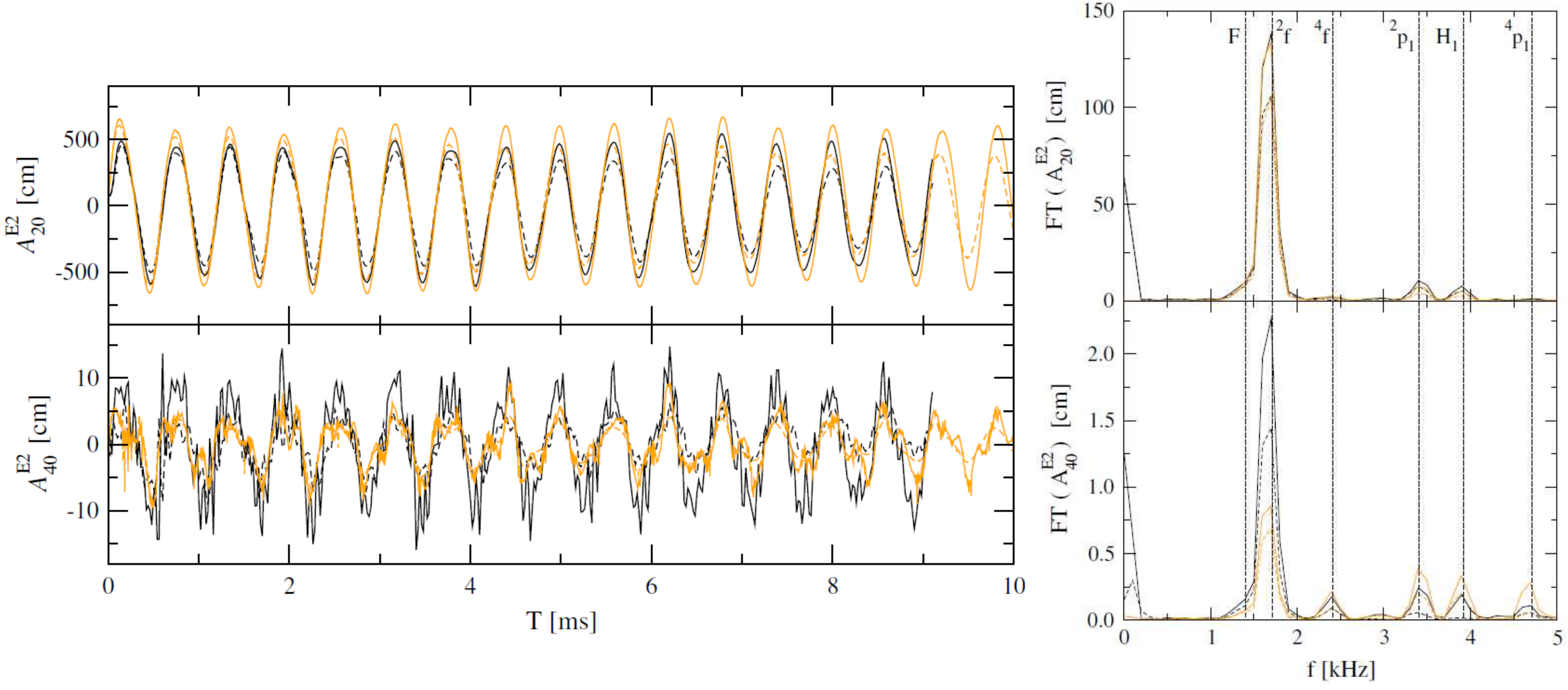
Expected order of convergence.

FCF: hyperbolic system and gravitational radiation

- Perturbed equilibrium rotating neutron stars:

Gravitational waves computed using Weyl scalars, taking into account off-sets.

Quadrupole and hexadecapole components (black lines) compared to post-Newtonian estimations (orange):



PIRK (Partially Implicit Runge Kutta) methods

- Motivation: Wave-like equation in spherical coordinates (hyperbolic equations which encode the gravitational radiation).

- Structure of the system of equations:

$$\begin{cases} u_t = \mathcal{L}_1(u,v) \\ v_t = \mathcal{L}_2(u) + \mathcal{L}_3(u,v) \end{cases} \longrightarrow \quad L_1, L_2, L_3$$

* u is evolved explicitly; v is evolved taking into account the updated value u for the evaluation of the L2 operator.

* No analytical/numerical inversion: costs comparable to explicit methods.

- Derived up to third-order in such a way that the number of stages is minimized.

- Recovery of the optimal SSP explicit RK methods when implicitly treated parts are neglected.

- Stability analysis to derive the remaining coefficients.

PIRK (Partially Implicit Runge Kutta) methods

- First order:

$$\begin{cases} u^{n+1} = u^n + \Delta t\, L_1(u^n, v^n), \\ v^{n+1} = v^n + \Delta t\left[(1-c_1)L_2(u^n) + c_1 L_2(u^{n+1}) + L_3(u^n, v^n)\right] \end{cases} \longrightarrow \quad c_1 = 1$$

- Second order:

$$\begin{cases} u^{(1)} = u^n + \Delta t\, L_1(u^n, v^n), \\ v^{(1)} = v^n + \Delta t\left[(1-c_1)L_2(u^n) + c_1 L_2(u^{(1)}) + L_3(u^n, v^n)\right], \end{cases}$$

$$\begin{cases} u^{n+1} = \frac{1}{2}\left[u^n + u^{(1)} + \Delta t\, L_1(u^{(1)}, v^{(1)})\right], \\ v^{n+1} = v^n + \frac{\Delta t}{2}\left[L_2(u^n) + 2c_2 L_2(u^{(1)}) + (1-2c_2)L_2(u^{n+1}) \right. \\ \qquad \left. + L_3(u^n, v^n) + L_3(u^{(1)}, v^{(1)})\right]. \end{cases}$$

$$\longrightarrow \quad (c_1, c_2) = (1/2, 0)$$

$$(c_1, c_2) = (1 - \sqrt{2}/2, (\sqrt{2}-1)/2)$$

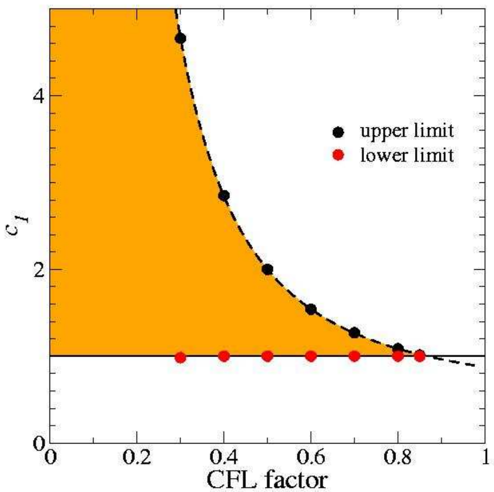
PIRK methods: scalar wave equation

$$\partial_{tt} h = \Delta h \quad \longrightarrow \quad \begin{cases} \partial_t h = A \\ \partial_t A = \Delta h \end{cases}$$

The system has solutions of the form $h(r, \theta, \varphi, t) \sim j_n(kr) Y_{lm}(\theta, \varphi) \cos \omega t$, being the first factor the spherical Bessel function, the second factor the corresponding spherical harmonic, and |k|=|w|.

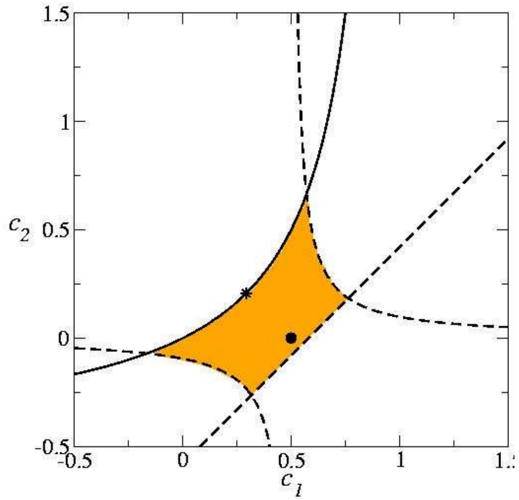
Spherical coordinates. Simulations inside a sphere of r=1, imposing zero values at r=1, with n=1. (l, m) = (0,0), (2,0), (2,2) for 1D, 2D and 3D cases.

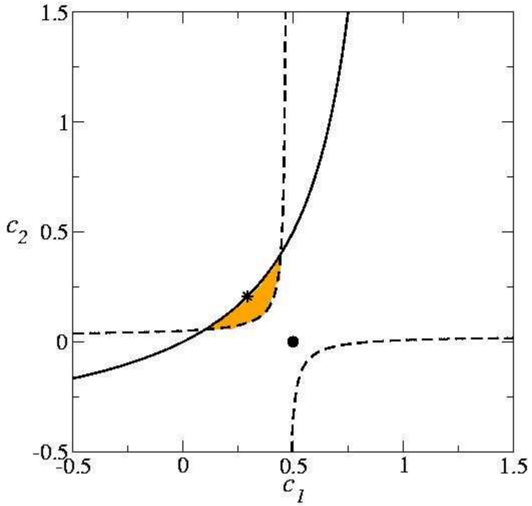
- First order: $c_1 = 1$



- Second order: $(c_1, c_2) = (1/2, 0)$

$(c_1, c_2) = (1 - \sqrt{2}/2, (\sqrt{2} - 1)/2)$





PIRK methods applied to Einstein equations

- PIRK methods were used to evolved the hyperbolic sector in the FCF using spherical coordinates without numerical instabilities. This sector encodes the gravitational radiation of the system. FCF belongs to the constrained ones.

- What happens in free formulations? In BSSN, the hyperbolic equations encode the gravitational radiation and also include the evolution of gauge variables.

Covariant formulation by Brown (2009). Simulations with spherical coordinates:

* Specific gauge choice: polar (Bardeen and Piran, 1983) or areal (Choptuik, 1991)… restricted gauge.

* Regularization procedure (Rinne ans Steward, 2005; Alcubierre and González, 2005)… not easy to implement numerically, auxiliary variables and their evolution equations required, difficult in the 3D case.

* Spherical reduction (Garfinkle et al, 2008; Bernuzzi and Hilditch, 2010)… simulations only in the spherical case.

* PIRK methods (computational costs similar to explicit methods) for reduced 1D BSSN equations (Montero and C.-C., 2012).

PIRK methods applied to Einstein equations

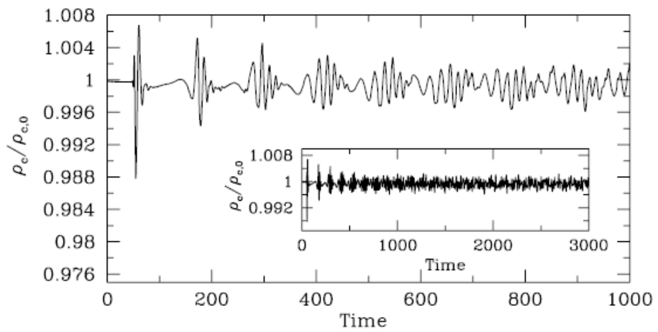
Montero and C.-C., 2012:

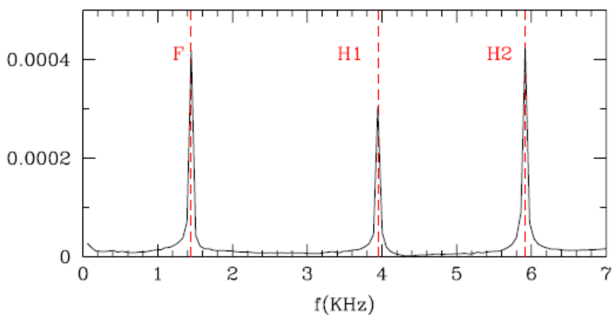
- Pure gauge pulse.

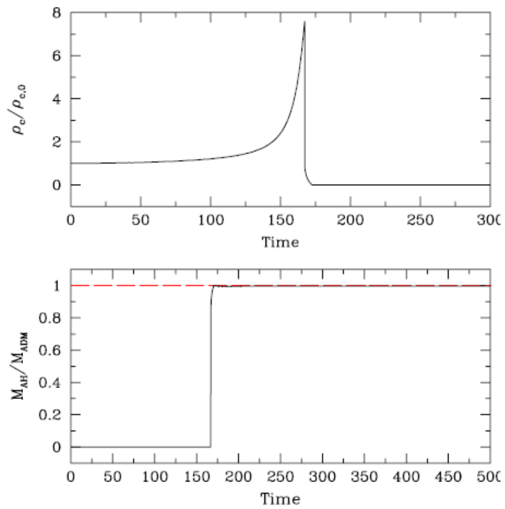
- Evolution of Schwarzschild BH (isotropic coordinates and moving puncture).

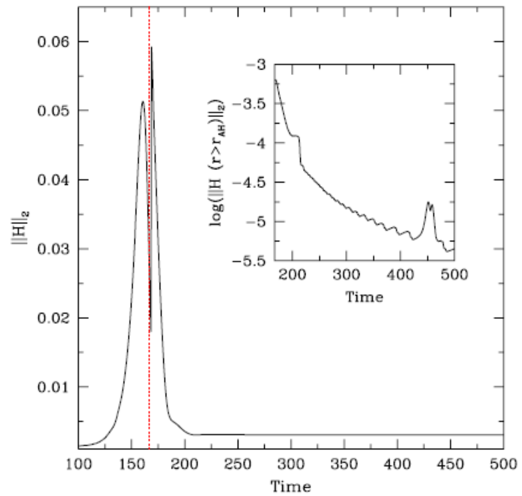
- Evolution of TOV equilibrium model.

- Gravitational collapse of marginally stable spherical relativistic star.









The PIRK methods also work in the case of 3D BSSN equations in spherical coordinates (in preparation).

Conclusions and future work

· A relativistic treatment is necessary to study compact objects and other astrophysical scenarios.

· 3+1 formalism: free evolution versus constrained formulations.

· Constrained formulations: from CFC to FCF: addition of the gravitational radiation.

· PIRK methods and its application to hyperbolic equations in some formulations.

· Work in progress:

* Coupling of equations in FCF. Excision method for elliptic equations.

* 3D simulations in spherical coordinates in BSSN.

Realistic comparison between BSSN and FCF in curvilinear coordinates (PIRK methods).